

Introduction to vectors

Notation: \mathbb{R} = the real numbers

\mathbb{R}^2 = the set $\{(a, b) : a, b \in \mathbb{R}\}$ of ordered pairs of real numbers
= “the (Euclidean) plane,” or “Euclidean 2-space”
(relative to a coordinate system)

\mathbb{R}^3 = the set $\{(a, b, c) : a, b, c \in \mathbb{R}\}$ of ordered triples of real numbers
= “Euclidean 3-dimensional space” (relative to a coordinate system)

\mathbb{R}^n = the set $\{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{R}\}$ of ordered n -tuples of real numbers
= “Euclidean 3-dimensional space” (relative to a coordinate system)

Goal: Establish mathematical object representing *displacements* in \mathbb{R}^2 , \mathbb{R}^3 , or even \mathbb{R}^n

Definition; Text: A *vector* in the plane is a pair $\langle a, b \rangle \in \mathbb{R}^2$

Similarly, a vector in 3-space is a triple $\langle a, b, c \rangle \in \mathbb{R}^3$

More generally, an *n-dimensional vector* is an n-tuple $\langle u_1, u_2, \dots, u_n \rangle \in \mathbb{R}^n$

Note: we use the $\langle a, b \rangle$ notation instead of (a, b) to emphasize the *role* of the tuple.

Components: The values u_1, u_2, \dots are called the *components* of $\langle u_1, u_2, \dots, u_n \rangle$

Warning: A vector, as a conceptual object, should exist independently of coordinate system. (Example in class.)

To say that the vector *is* this n-tuple is therefore not quite correct.

A vector is an ‘ideal representation’ of a displacement in the plane (space, etc.), which has *magnitude* and *direction*.

It is more honest to say:

We *assume* some coordinate system is fixed, and then ‘identify’ the space of n-dimensional vectors with \mathbb{R}^n

Can describe vectors pictorially (examples - class)

Operations.

Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and $c \in \mathbb{R}$ (a *scalar*)

Vector addition: $\mathbf{u} + \mathbf{v} := \langle u_1 + v_1, u_2 + v_2 \rangle$

Scalar multiplication: $c\mathbf{u} := \langle cu_1, cu_2 \rangle$

Examples (class)

These are the two operations that characterize an *abstract vector space* (Math 415)

Other operations can be defined in terms of these, eg:

$$-\mathbf{u} := (-1)\mathbf{u}$$

$$\mathbf{u} - \mathbf{v} := \mathbf{u} + (-\mathbf{v})$$

More generally we can define the same operations for \mathbb{R}^n , $n \geq 3$:

Let $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$, $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$

Vector addition: $\mathbf{u} + \mathbf{v} := \langle u_1 + v_1, u_2 + v_2, \dots, u_n + v_n \rangle$

Scalar multiplication: $c\mathbf{u} := \langle cu_1, cu_2, \dots, cu_n \rangle$

Other operations, and special vectors.

Equality: $\langle u_1, u_2, \dots, u_n \rangle = \langle v_1, v_2, \dots, v_n \rangle$ provided $u_i = v_i$ for all i

Zero vector: $\mathbf{0} := \langle 0, 0, \dots, 0 \rangle$. Note that there is a different zero vector for each \mathbb{R}^n .

Standard component vectors: (In \mathbb{R}^2) $\mathbf{i} := \langle 1, 0 \rangle$, $\mathbf{j} := \langle 0, 1 \rangle$.

(In \mathbb{R}^3) $\mathbf{i} := \langle 1, 0, 0 \rangle$, $\mathbf{j} := \langle 0, 1, 0 \rangle$, $\mathbf{k} := \langle 0, 0, 1 \rangle$.

Directions: \mathbf{u} and \mathbf{v} *have the same direction* if $\mathbf{u} = c\mathbf{v}$ for some $c > 0$

\mathbf{u} and \mathbf{v} *have opposite directions* if $\mathbf{u} = c\mathbf{v}$ for some $c < 0$

\mathbf{u} and \mathbf{v} are *parallel* if $\mathbf{u} = c\mathbf{v}$ or $\mathbf{v} = c\mathbf{u}$ for some $c \in \mathbb{R}$

Length: $|\langle u_1, u_2, \dots, u_n \rangle|$ = the *length* or *norm* of $\langle u_1, u_2, \dots, u_n \rangle$
 $:= \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$

(Sometimes you'll see $\|\mathbf{u}\|$ instead of $|\mathbf{u}|$.)

Examples (class)

Basic algebraic properties of vector operations

Theorem: Vector addition is commutative and associative,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

Multiplication by scalars is associative,

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

and satisfies the two distributive laws

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

(Here c, d are scalars, \mathbf{u}, \mathbf{v} are vectors.)