MATH251a, Fall '08
Exam 1
Prof. Ross's Section

Name: Sol'ns

INSTRUCTIONS: Write legibly. Indicate your answer clearly in the space provided. Show all work; explain your answers. Answers with work not shown might be worth zero points. You may only use a calculator as provided in class, and may use one 8×11 'crib sheet'. Cheating is *not* permitted.

Problem	Worth	Score
1	15	
2	10	
3	15	
4	10	
5	40	
6	10	
7	10	
Total	115	

(15) 1. Let $\mathbf{u} = \langle 2, -3\rangle$ and $\mathbf{v} = \langle 5, -1\rangle$. Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\mathbf{u} \cdot \mathbf{v}$, and the vector projection of $\mathbf{u}$ onto $\mathbf{v}$

$$\|\underset{\sim}{u}\| = \sqrt{2^2+(-3)^2} = \sqrt{4+9} = \sqrt{13}$$

$$\|\underset{\sim}{v}\| = \sqrt{5^2+(-1)^2} = \sqrt{25+1} = \sqrt{26}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = 2\cdot 5 + (-3)(-1) = 10+3 = 13$$

$$\overrightarrow{\text{proj}}_{\underset{\sim}{v}}\underset{\sim}{u} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{v}\|^2}\underset{\sim}{v} = \left(\frac{13}{26}\right)\langle 5,-1\rangle = \langle \tfrac{5}{2}, -\tfrac{1}{2}\rangle$$

(15) 2. Find all values of x such that $\underbrace{\langle 1, x\rangle}_{\underset{\sim}{u}} \perp \underbrace{\langle 1-2x, x\rangle}_{\underset{\sim}{v}}$

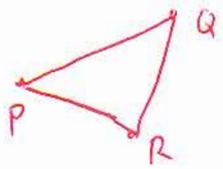
$$\underset{\sim}{u} \perp \underset{\sim}{v} \iff \underset{\sim}{u} \cdot \underset{\sim}{v} = 0$$

$$\iff 1\cdot(1-2x) + x\cdot x = 0$$

$$\iff \underbrace{x^2-2x+1}_{(x-1)^2} = 0$$

$$\iff x = 1$$

(15) 3. Let $\mathbf{P}(1,1), \mathbf{Q}(3,5), \mathbf{R}(2,-5)$ be the vertices of a triangle. Find the *cosines* of the three angles in the triangle. (No need to find the actual angles.)



$$\cos \angle QPR = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\,\|\overrightarrow{PR}\|} = \frac{\langle 3-1, 5-1\rangle \cdot \langle 2-1, -5-1\rangle}{\|\langle 2,4\rangle\|\;\|\langle 1,-6\rangle\|}$$

$$= \frac{\langle 2,4\rangle\cdot\langle 1,-6\rangle}{\sqrt{2^2+4^2}\,\sqrt{1^2+(-6)^2}} = \frac{2-24}{\sqrt{20}\,\sqrt{37}} = \frac{-22}{\sqrt{20}\sqrt{37}}$$

similarly,

$$\cos\angle PQR = \frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{\|\overrightarrow{QP}\|\,\|\overrightarrow{QR}\|} = \frac{\langle -2,-4\rangle\cdot\langle -1,-10\rangle}{\sqrt{20}\,\sqrt{101}} = \frac{42}{\sqrt{20}\sqrt{101}}$$

$$\cos\angle PRQ = \frac{\overrightarrow{RP}\cdot\overrightarrow{RQ}}{\|\overrightarrow{RP}\|\,\|\overrightarrow{RQ}\|} = \frac{\langle -1,6\rangle\cdot\langle 1,10\rangle}{\sqrt{37}\,\sqrt{101}} = \frac{59}{\sqrt{37}\sqrt{101}}$$

(10) 4. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^2$ then $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$

Let $\underset{\sim}{a} = \langle a_1, a_2\rangle \quad \underset{\sim}{b} = \langle b_1, b_2\rangle$

$\underset{\sim}{a} + \underset{\sim}{b} = \langle a_1, a_2\rangle + \langle b_1, b_2\rangle$

$= \langle a_1+b_1, a_2+b_2\rangle$ (by def of vector addition)

$= \langle b_1+a_1, b_2+a_2\rangle$ (by commutative law of addition on $\mathbb{R}$)

$= \langle b_1, b_2\rangle + \langle a_1, a_2\rangle$ (def of vector +)

$= \underset{\sim}{b} + \underset{\sim}{a}$

(40) 5. Find the indicated limits, or show thay do not exist. Justify your answer (but do *not* give a formal proof involving ϵ, δ).

(a) $\lim_{x\to 3}(3x+1)(2x-5)^{2008} = ?$

Since $(3x+1)(2x-5)^{2008}$ is a polynomial (of degree 2009),

$$\lim_{x\to 3}(3x+1)(2x-5)^{2008} = (3\cdot 3+1)(2\cdot 3-5)^{2008} = (10)(1)^{2008} = 10$$

(b) $\lim_{t\to 2}\frac{t^2-4}{(t+1)^2-9} = ?$

$$= \lim_{t\to 2}\frac{(t+2)\cancel{(t-2)}}{(t+1+3)\cancel{(t+1-3)}} = \lim_{t\to 2}\frac{t+2}{t+4} = \frac{2+2}{2+4} = \frac{4}{6}$$

since $\frac{t+2}{t+4}$ is a rational function

(c) $\lim_{x\to\infty}\frac{3x^2-12}{2x^2+x+1} = ?$

$$= \lim_{x\to\infty}\frac{3-12/x^2}{2+1/x+1/x^2} = \frac{3}{2}$$

(d) $\lim_{t\to 0}\frac{|t|}{t} = ?$

$$\frac{|t|}{t} = \begin{cases} t/t & \text{if } t>0 \\ -t/t & \text{if } t<0 \end{cases} = \begin{cases} 1 & \text{if } t>0 \\ -1 & \text{if } t<0 \end{cases}$$

$$\therefore \lim_{t\to 1^-}\frac{|t|}{t} = -1 \neq 1 = \lim_{t\to 1^+}\frac{|t|}{t}$$

$\therefore$ limit does not exist.

(10) 6. Prove that $\lim_{x\to 0} x^2 \sin \frac{1}{x} = 0$

Since $-1 \le \sin\frac{1}{x} \le 1$ for any x $(x \neq 0)$,

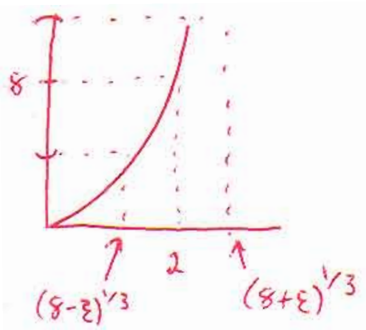
$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$ for any $x \neq 0$.

Also, $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$, so

by the squeeze theorem,

$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

(15) 7. Prove (using a $\delta - \epsilon$ argument) that $\lim_{x\to 2} x^3 = 8$



Let $\varepsilon > 0$; we may suppose that $\varepsilon < 8$.

Let δ be whichever is smaller, $2 - \sqrt[3]{8-\varepsilon}$ and $\sqrt[3]{8+\varepsilon} - 2$.

Then, if $0 < |x-2| < \delta$

Then $2-\delta < x < 2+\delta$

$\therefore \sqrt[3]{8-\varepsilon} < x < \sqrt[3]{8+\varepsilon}$

$\therefore 8-\varepsilon < x^3 < 8+\varepsilon$ (since $f(x)=x^3$ is an increasing function)

$\therefore -\varepsilon < x^3 - 8 < \varepsilon$

$\therefore |x^3 - 8| < \varepsilon$. This proves the limit.