# Exam 3 Sol'ns

(10) 1. $\sum_{i=1}^{5}(i+1)(i+2) =$? More generally, $\sum_{i=1}^{n}(i+1)(i+2) =$? (Your answer to this second part should be an algebraic formula in $n$.)

$$\sum_{i=1}^{5}(i+1)(i+2) = 2\cdot3+3\cdot4+4\cdot5+5\cdot6+6\cdot7$$

$$\sum_{i=1}^{n} i^2+3i+2 = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2$$

$$= \frac{n(n+1)(2n+1)}{6} + 3\frac{n(n+1)}{2} + 2n$$

(15) 2. Exactly how many real solutions does the equation $100x^3 + x - 1 = 0$ have? Why? (You need not – in fact *should* not, and probably *can* not – actually solve this equation!)

Put $f(x) = 100x^3+x-1$; $f$ is continuous & differentiable everywhere since it is a polynomial

$f(0) = -1 < 0$, $f(1) = 100 > 0$, $\therefore$ by IVT there is at least one sol'n to $f(x)=0$

$f'(x) = 300x^2+1$ always $> 0$, so there is at most 1 sol'n to $f(x)=0$

$\therefore$ this eq'n has exactly one sol'n

(10) 3. Suppose $f$ is a differentiable function, that $f(1) = 2$, and that $f'(x) < 10$ when $1 < x < 5$. How big can $f(5)$ be?

$f(5) - f(1) = f'(c)(5-1)$ for some $c \in (1,5)$ by MVT

$\therefore f(5) = f(1) + f'(c)\cdot4 = 2+4f'(c) \le 2+4|f'(c)| < 2+4\cdot10 = \underline{\underline{42}}$

(15) 4. What are the maximum and minimum values of $f(t) = \frac{t+1}{t^2+t+1}$ on the interval $[-3, 1]$?

$$f'(t) = \frac{(t^2+t+1)-(t+1)(2t+1)}{(t^2+t+1)^2} = \frac{-t^2-2t}{(t^2+t+1)^2}$$

Set $=0$, $-t(t+2)=0$ $\therefore t=0$ or $t=-2$

| | $t$ | $f(t)$ | |
|---|---|---|---|
| | 0 | 1 | ← max |
| | −2 | −1/3 | ← min |
| endpts | 1 | 2/3 | |
| endpts | −3 | −2/7 | |

$f$ has a min of $-1/3$ at $t=-2$

$f$ has a max of $1$ at $t=0$

5. Find a function $f(x)$ satisfying $f'(x) = -3\cos^2 x \sin x$, $\quad f(\pi/4) = 1/2$

$f(x) = \cos^3 x + K$ works

$f(\pi/4) = 1/2 \Rightarrow 1/2 = (\sqrt{2}/2)^3 + K = \frac{2\sqrt{2}}{8} + K \quad \therefore K = \frac{1}{2} - \frac{\sqrt{2}}{4} = \frac{2-\sqrt{2}}{4}$

$\therefore f(x) = \cos^3 x + \frac{2-\sqrt{2}}{4}$

6. Consider the function $f(x) = \frac{x^2-4}{x^3}$. (a) Find $f'(x), f''(x)$ (b) On which intervals is this function increasing? Decreasing? Concave up? Concave down? Find: (c) all local and global extrema, (d) any inflection points. (e) Are there asymptotes? (f) Graph the function.

(a) $f(x) = x^{-1} - 4x^{-3} \quad \therefore f'(x) = -x^{-2} + 12x^{-4} = \frac{12-x^2}{x^4}$, $f''(x) = 2x^{-3} - 48x^{-5} = \frac{2x^2-48}{x^5}$

(b) $f'(x) = 0 \iff x = \pm\sqrt{12} = \pm 2\sqrt{3}$

$f''(x) = 0 \iff x^2 = 24 \iff x = \pm 2\sqrt{6}$

$f''$ : - - 0 + + + + + + u - - - - - - - 0 + + +

$f'$ : - - - - - 0 + + + + u + + + 0 - - - - -

$-2\sqrt{6}$ $\quad -2\sqrt{3}$ $\quad 0$ $\quad 2\sqrt{3}$ $\quad 2\sqrt{6}$

Increasing on $[-2\sqrt{3}, 0)$ and $(0, 2\sqrt{3}]$

decreasing on $(-\infty, -2\sqrt{3}]$ and $[2\sqrt{3}, \infty)$

Conc. up on $(-2\sqrt{6}, 0)$ and $(2\sqrt{6}, \infty)$

" down on $(-\infty, -2\sqrt{6})$ and $(0, 2\sqrt{6})$

(c) Local min at $x = -2\sqrt{3}$, local max at $x = 2\sqrt{3}$

No global min/max (see e)

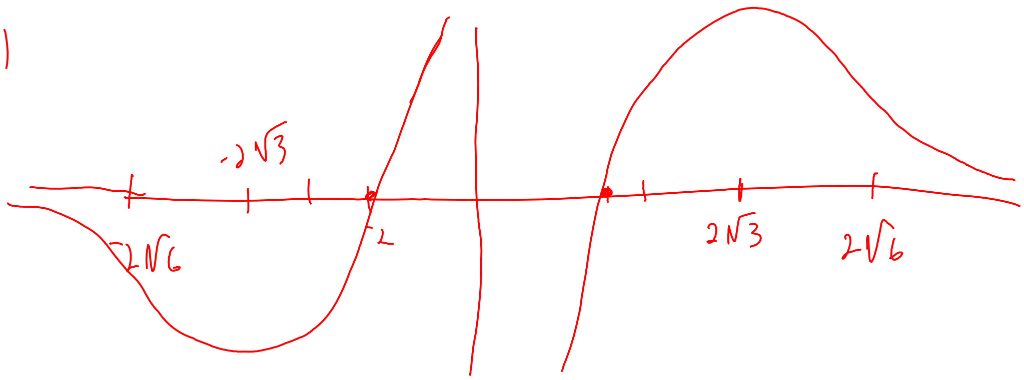
(d) Yes - concavity changes at $\pm 2\sqrt{6}$

(e) $\lim_{x \to \pm\infty} f(x) = 0$, so horizontal asymptote of $y = 0$

$\lim_{x \to 0^-} f(x) = \infty$, $\lim_{x \to 0^+} f(x) = -\infty$, so vertical asymptote at $x = 0$

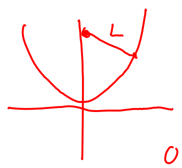
(also: no abs. min or max)

(f)




7. Do ONE of the following TWO problems. Note that the second is worth more than the first; it is a bit harder. Indicate CLEARLY which one you want counted.

(a) (15) Find the point on the parabola $x^2 = 4y$ nearest the point $(0, 5)$ on the $y$-axis.




Min $L = \sqrt{x^2 + (y-5)^2} = \sqrt{4y + (y-5)^2}$

s.t. $y \geq 0$

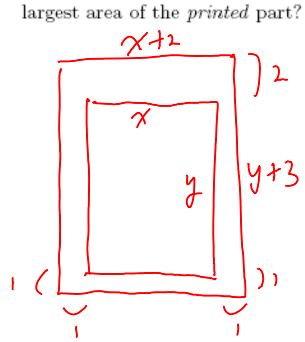
OR min $Z = L^2 = 4y + (y-5)^2$

$Z' = 4 + 2(y-5)$ set $= 0$, $y = \frac{-4}{2} + 5 = 3$ (and $x = \pm\sqrt{12}$)

$Z'' = 2 > 0$, so this is a local min. Since it is the only critical point, it is a global min.

$\therefore$ distance is minimized at $(\pm\sqrt{12}, 3)$

(b) (20) A rectangular poster is to have a total area of 180 square inches, with 1-inch margins at the sides and the bottom and a 2-inch margin at the top. What dimensions of the poster give the largest area of the *printed* part?




Maximize $A$ = print area $= xy$

where $(x+2)(y+3) = 180$

$0 \leq x, y$

Note $y = \frac{180}{x+2} - 3$, and $y \geq 0 \Rightarrow \frac{180}{x+2} \geq 3$, $\therefore x \leq 58$

So Maximize $A = (x)\left(\frac{180}{x+2} - 3\right) = \frac{180x}{x+2} - 3x$

s.t. $0 \leq x \leq 58$

$A' = \frac{(x+2)(180) - 180x}{(x+2)^2} - 3 = \frac{360}{(x+2)^2} - 3$ set $= 0$,

$(x+2)^2 = 120$, $x + 2 = \pm\sqrt{120}$ so $x = \sqrt{120} - 2$ (otherwise $x < 0$)

If $x = 0$ or $58$ then $A = 0$

If $x = \sqrt{120} - 2$ then $A = (\sqrt{120} - 2)\left(\frac{180}{\sqrt{120}} - 3\right) > 0$, so this is the maximum:

$x = \sqrt{120} - 2$, $y = \frac{180}{\sqrt{120}} - 3$