MATH252a, Spring '09 **Your Name:**
Exam 3
Prof. Ross's Lecture

INSTRUCTIONS: Write legibly. Indicate your answer clearly in the space provided. Show all work; explain your answers. Answers with work not shown might be worth **zero** points. You may NOT use a calculator (unless I provide one), but MAY use one 8×11 'crib sheet' (just one side please!). Cheating is *not* permitted.

Page	Points	Score
1	15	
2	50	
3	40	
4	40	
5	30	
Total:	175	

(15) 1. Let $\mathcal{C}$ be the curve given by $y = 1 + x^2, 1 \leq x \leq 3$. Set up integrals - DO NOT INTEGRATE - representing the (a) the length of $\mathcal{C}$, and (b) the surface area if $\mathcal{C}$ is rotated around the $x-$axis. (5 points extra credit if you also give an integral for the surface area of the shape obtained by rotating $\mathcal{C}$ around the line $y = 25$.)

$y' = 2x$

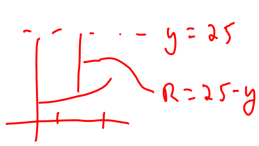
$ds = \sqrt{1 + (y')^2}\, dx$

$= \sqrt{1 + 4x^2}\, dx$

a) $L = \int ds = \int_1^3 \sqrt{1 + 4x^2}\, dx$

b) $SA = \int 2\pi R\, ds = \int 2\pi y\, ds = \int_1^3 2\pi (1 + x^2)\sqrt{1 + 4x^2}\, dx$

EC



$\therefore SA = \int_1^3 2\pi (25 - y)\, ds = \int_1^3 2\pi (24 - x^2)\sqrt{1 + 4x^2}\, dx$

(15) 2. Compute the indicated limits.

(a) $\lim_{n\to\infty}(1+\frac{3}{n})^{2n}$ $= \left[\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n\right]^2 = (e^3)^2 = e^6$

(b) $\lim_{n\to\infty} 2\ln 3n - \ln(n^2-5n+7)$ $= \lim_{n\to\infty} \ln\left[\underbrace{\frac{(3n)^2}{n^2-5n+7}}_{\to 9}\right] = \ln 9$

(15) 3. Rewrite $4.001212121212\ldots$ as an infinite series, and put in the form $A+\frac{P}{Q}$ where A, P and Q are integers.

$= 4 + \frac{12}{10^4} + \frac{12}{10^6} + \cdots = 4 + \sum_{n=2}^{\infty} \frac{12}{10^{2n}} = 4 + \frac{12/10^4}{1-10^2} = 4 + \frac{12}{9900}$

(20) 4. Compute the indicated sum.

(a) $\sum_{n=1}^{\infty}\frac{2^{2n}}{9^n}$ $= \frac{4}{9} + \frac{4^2}{9^2} + \cdots = \frac{4/9}{1-4/9} = \frac{4}{5}$ (Geo series $a = 4/9$, $r = 4/9$)

(b) $\sum_{n=1}^{\infty}\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}$ (Hint: )

Telescoping series:

$S_N = \left(\frac{1}{1^{3/2}} - \cancel{\frac{1}{2^{3/2}}}\right) + \left(\cancel{\frac{1}{2^{3/2}}} - \cancel{\frac{1}{3^{3/2}}}\right) + \cdots + \left(\cancel{\frac{1}{N^{3/2}}} - \frac{1}{(N+1)^{3/2}}\right)$

$= 1 - \frac{1}{(N+1)^{3/2}} \to 1$ as $N \to \infty$

$\therefore$ Series converges to 1

(20) 5. Test the series $\sum_{k=2}^{\infty} \overbrace{(-1)^k \frac{1}{\sqrt{k}-1}}^{a_k}$ for (a) absolute and (b) conditional convergence. (c) How many terms would we need to compute the sum with an error of at most $\frac{1}{99}$?

a) $|a_k| = \frac{1}{\sqrt{k}-1} > \frac{1}{\sqrt{k}}$, and $\sum_{k=2}^{\infty} \frac{1}{k^{1/2}}$ diverges (p-test, $p = 1/2 < 1$)

$\therefore$ series does not converge absolutely.

b) ① $\lim_{k\to\infty} \frac{1}{\sqrt{k}-1} = 0$, and ② since $\sqrt{k}$ is an increasing function, $\frac{1}{\sqrt{k}-1}$ is a decreasing function

③ Series is alternating

$\therefore$ Series converges conditionally, by 3nd test

c) error $< |a_k|$, put $\frac{1}{\sqrt{k}-1} \leq \frac{1}{99}$, $\sqrt{k}-1 \geq 99$, $\sqrt{k} \geq 100$, $k = 10^4$

(so $10^4 - 1$ terms)

(20) 6. Find the radius of convergence and interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$

root test $\sqrt[n]{|a_n|} = \frac{|x|}{(\sqrt[n]{n})2} \to \frac{|x|}{2}$ as $n \to \infty$

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converges if $\frac{|x|}{2} < 1$ or $|x| < 2$

diverges if $|x| > 2$

test endpoints:

$x = 2$ $\sum \frac{2^n}{n \cdot 2^n} = \sum \frac{1}{n}$ diverges (Harmonic series)

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$x = -2$ $\sum \frac{(-2)^n}{n \cdot 2^n} = \sum \frac{(-1)^n}{n}$ converges (alternating harmonic)

$\therefore$ ROC $= 2$, IOC $= [-2, 2)$

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(40) 7. Test the series for convergence or divergence. Do not attempt to actually compute!

(a) $\sum_{n=1}^{\infty} \underbrace{\left(\frac{n+1}{2n}\right)^n}_{a_n}$

$\sqrt[n]{a_n} = \frac{n+1}{2n} \to \frac{1}{2}$ as $n \to \infty$

$\frac{1}{2} < 1$, so $\sum a_n$ converges by root test

(b) $\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{n+2}}_{a_n}$

$a_n \to 1 \neq 0$, so series diverges by bad test

(c) $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2+n}}_{a_n}$

$a_n < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges (p-test, $p=2>1$)

so series converges by comparison test

(d) $\sum_{n=0}^{\infty} \underbrace{\frac{2^n n!}{n^n}}_{a_n}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = \frac{2(n+1)\,n^n}{(n+1)^{n+1}}$$

$$= 2\left(\frac{n}{n+1}\right)^n = \frac{2}{(1+1/n)^n} \to \frac{2}{e} \text{ as } n \to \infty.$$

Since $\frac{2}{e} < 1$, series converges

by ratio test

(30) 8. Do TWO of the following THREE problems. Indicate clearly which you are doing.

(a) A sequence is defined recursively by $a_1 = 2; a_{n+1} = \frac{1}{3-a_n}$ (i) Write out the first 5 terms of this sequence. (ii) Assuming that $L = \lim_{n\to\infty} a_n$ exists, find L.

(b) If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = \frac{n-1}{n+1}$ then (i) find a_n and (ii) find $\sum_{n=1}^{\infty} a_n$

(c) Show that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. Do NOT use the Hierarchy Theorem.

a) $2, 1, \frac{1}{2}, \frac{1}{3-1/2} = \frac{2}{5}, \frac{1}{3-2/5} = \frac{5}{13}$. If $\lim_{n\to\infty} a_n = L$,

then $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n} = \frac{1}{3-L}$, so $3L - L^2 = 1$

$\therefore L^2 - 3L + 1 = 0$, $L = \frac{3 \pm \sqrt{9-4}}{2}$, $L = \frac{3+\sqrt{5}}{2}$ (negative root not possible)

b) $S_{n-1} + a_n = S_n$, so $a_n = S_n - S_{n-1} = \frac{n-1}{n+1} - \frac{n-2}{n}$

$\sum a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n-1}{n+1} = 1$

c) $\frac{2^n}{n!} = \frac{2\cdot 2\cdot 2\cdot \cdots \cdot 2}{1\cdot 2\cdot 3\cdots n} \le 2\cdot 1\cdot\left(\frac{2}{3}\right)^{n-2} \to 0$ as $n\to\infty$, so $\frac{2^n}{n!} \to 0$

OR use ratio test to show that $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges,

then $\frac{2^n}{n!} \to 0$ as $n\to\infty$ by nth term test.