Comparison Tests

Suppose $\{a_n\}_n$, $\{b_n\}_n$ are nonnegative sequences.

(a) If $\sum b_n$ converges and $a_n \le b_n$ for all sufficiently large n,

Then $\sum a_n$ converges

(b) If $\sum b_n$ diverges and $b_n \le a_n$ for all sufficiently large n,

Then $\sum a_n$ diverges

<u>EG</u> $\sum \frac{n^2}{2n^4+n+1}$ Let $a_n = \frac{n^2}{2n^4+n+1}$, $b_n = \frac{1}{2n^2}$

note $a_n = \frac{1}{2n^2+\frac{1}{n}+\frac{1}{n^2}} \le \frac{1}{2n^2} = b_n$

also, $\sum b_n = 2 \sum \frac{1}{n^2}$ converges (p-test, $p \geq 2 > 1$)

so $\sum a_n$ converges.

<u>EG</u> $\sum \frac{n^2}{2n^4-n+1}$, let $a_n = \frac{n^2}{2n^4-n+1}$, <u>can't use</u> $\frac{1}{2n^2}$ for b_n

Since $a_n = \frac{1}{2n^2-\frac{1}{n}+\frac{1}{n^2}} > b_n$, and being <u>bigger</u> than

a convergent series <u>doesn't</u> <u>show</u> <u>anything</u>!

<u>But</u> for n large, $a_n \le \frac{1}{2n^2-1+0} \le \frac{1}{2n^2-n^2} = \underbrace{\frac{1}{n^2}}_{b_n}$

EG $\sum_{n=3}^{\infty} \underbrace{\frac{n+\sin n}{n^2-5}}_{a_n}$ would like to compare to $\frac{1}{n}$, so guess: diverges

$$\frac{n+\sin n}{n^2-5} = \frac{1+\sin(n)/n}{n-5/n} \geq \frac{1+\sin(n)/n}{n}$$

$$= \frac{\frac{1}{2} + \left(\frac{1}{2} + \frac{\sin n}{n}\right)}{n} \geq \frac{1}{2n} \leftarrow b_n$$

Since $b_n \leq a_n$ and $\sum b_n = \frac{1}{2}\sum\frac{1}{n}$ diverges (harmonic series)

Limit comparison test (makes difficult comparisons easier)

As before, suppose $\{a_n\}_n$ and $\{b_n\}_n$ nonnegative sequences

If $\frac{a_n}{b_n} \to L$ and $0 < L < \infty$ Then

either both $\sum a_n$ and $\sum b_n$ converge

or both $\sum a_n$ and $\sum b_n$ diverge

Proof Text & class

EG In last example, let $b_n = 1/n$; know $\sum b_n$ diverges.

$$\frac{a_n}{b_n} = \left(\frac{n+\sin n}{n^2-5}\right) \div \left(\frac{1}{n}\right) = \frac{n^2+n\sin n}{n^2-5} = \frac{1+\frac{\sin n}{n}}{1-\frac{5}{n}} \to 1 \text{ as } n\to\infty$$

since $0 < 1 < \infty$, $\sum a_n$ diverges