

Making sense of a^b

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In Section 4.1 our text defines the constant e as the only number a such that a^x is its own derivative. While this is a true *fact* about e , it is pretty convoluted: first they need to justify that functions of the form a^x make sense for x not a rational number, then they need to prove that a^x is differentiable, then they need to show that there really *is* a number a with this property.

In fact, they don't do any of these properly. They never even explain what a^x *means* when x is not a rational number! (They only say "it can be shown that" $\lim_{r \rightarrow x} a^r$ exists when x is not rational and r only takes rational values.

One important goal of mathematics is to make concepts like this clear and precise. Fortunately, we have enough tools already to do this. Even better, doing so will tie together several ideas you have already learned in this course.

Our strategy is this:

1. Give correct definitions for the functions e^x and $\ln x$. Of course, once we have one of these, the other is just the inverse (as long as we know that the functions are invertible).
2. Develop useful properties for these functions - in fact, the very rules that you've used for years for a^b and $\log_a b$, though in this case just for e and the natural logarithm.
3. *Define* a^x and $\log_a x$ in terms of e^x and $\ln x$; you probably already know how to do this.

The text finally does get around to doing this in Section 6.6, but we can do it now.

1 e^x

There are three common ways that e^x is defined:

1. $e^x = \lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n$ This definition has a nice interpretation in terms of interest rates: if we invest \$1.00 for 1 year at $100x\%$ interest (e.g., 4% interest means $x = 0.04$), and if we compound the interest n times each year, then the dollar's value at the end of the year is $(1 + \frac{x}{n})^n$. Some banks will compound *continuously*, which is just the limit as $n \rightarrow \infty$; in other words, the dollar is then worth e^x dollars.

The problem with this definition is that it is a bit difficult to prove that the limit even exists, let alone that the function e^x has the properties we want; I've only taught out of one Calculus text which takes this as the definition.

2. $e^x = 1 + x + \frac{x^2}{2 \times 1} + \frac{x^3}{3 \times 2 \times 1} + \frac{x^4}{4 \times 3 \times 2 \times 1} + \dots + \frac{x^n}{n!} + \dots$ (where $n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$). This is a very nice definition for obtaining some advanced properties of e^x , so is often used in graduate-level textbooks. However, to use it we need to define what we mean by the infinite "polynomial"; we won't do this in this course until the last part of the semester, so this will not be the definition we will use here.
3. $e^x = L^{-1}(x)$ where $L(x)$ is the natural logarithm function. While this method is less direct than the others, it is very straightforward and produces quite easily all the useful properties of both e^x and $\ln(x)$. This is the approach we will follow.

2 $\ln(x)$

We'll begin with the definition of the natural logarithm. We'll temporarily adopt a different name for the function, so that we don't accidentally make any assumptions about the it's behavior before we prove anything. So:

$$L(x) = \int_1^x \frac{1}{t} dt \text{ for } x > 0$$

Definition 2.1

We can immediately start drawing conclusions about the function $L(x)$. Most of them are so straightforward that I leave their proofs as exercises.

1. L is differentiable, with $L'(x) = \frac{1}{x}$ ($x > 0$)

Exercise 2.1 *Why? Also, why does this make L continuous?*

2. $L(1) = 0$; $L(x) > 0$ for $x > 1$; $L(x) < 0$ for $0 < x < 1$

Exercise 2.2 *Why?*

3. L is an increasing function (hence one-to-one, so invertible)

Exercise 2.3 *Why?*

4. L is concave down

Exercise 2.4 Why?

The next property is historically the reason Napier invented logarithms:

5. $L(ab) = L(a) + L(b)$ for any $a, b > 0$

To see why this is true, note that $L(ab) = \int_1^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt$ (WHY?) $= L(a) + \int_a^{ab} \frac{1}{t} dt$. We are therefore done if we can do the following exercise:

Exercise 2.5 Prove that $\int_a^{ab} \frac{1}{t} dt = \int_1^b \frac{1}{t} dt$ (which of course is just $L(b)$).

(Hint: Change of variables; let $u = t/a$, etc.)

Incidentally, well before the invention of the Calculus (and of course the invention of “integration by substitution”) a Belgian Jesuit named Gregory St. Vincent proved (basically) the following result: $L(ax) - L(ay) = L(x) - L(y)$ for $a, x, y > 0$. (He defined $L(x)$ in terms of areas instead of integrals.) This result easily implies Property ??; just take $x = b$, $y = 1$, and rearrange terms. Conversely, Property ?? implies St. Vincent’s result. (Show!) I can’t stress too highly how remarkable an achievement it was to prove this property without the machinery of the Calculus. The proof itself isn’t all that difficult, but the idea clearly presages many ideas we commonly attribute to Newton and Leibniz. (If I remember I will distribute separately a description of the proof, lifted from C. H. Edward’s nice text on the history of Calculus.)

The following stronger form is obtained by mathematical induction, or by iterating the last property n times. For example, $L(abc) = L((ab)c) = L(ab) + L(c) = L(a) + L(b) + L(c)$.

- 5'. $L(a_1 a_2 \dots a_n) = L(a_1) + \dots + L(a_n)$ for any $a_1, a_2, \dots > 0$

6. $L(\frac{1}{a}) = -L(a)$

This one is easy to see: $0 = L(1) = L(a \times \frac{1}{a}) = L(a) + L(\frac{1}{a})$, which is equivalent to the statement.

7. The range of L is all of \mathbb{R}

Since L is continuous, it suffices to show that L takes arbitrarily large positive and negative values (the Intermediate Value Theorem means that all other, intermediate values will be attained). Consider, for example, $L(2^n)$, which by Property 5' is $nL(2)$. Since $L(2) > 0$, $nL(2)$ does take arbitrarily large positive values (as n gets large). Of course, $L(2^{-n}) = nL(1/2) = -nL(2)$ takes arbitrarily large negative values too.

The next property will be used in the next section to show that e^x has the meaning we expect it to have.

8. $L(x^r) = rL(x)$ when r is a positive rational number

Recall that a *rational number* is a *ratio* of two integers, i.e. $r = \frac{m}{n}$ for integers m, n . Note that there is no assertion about x^r for *irrational* r , since we haven't even defined x^r for such r . Of course, when $r = \frac{m}{n}$, x^r is just the n^{th} root of x^m .

When r is an integer, this is an immediate consequence of Property 5': $L(x^r) = L(x + x + \cdots + x) = L(x) + L(x) + \cdots + L(x)$ (where the sums both have r terms) $= rL(x)$.

To see that Property 8 holds for more general $r = \frac{m}{n}$, just raise x^r to the n^{th} power and plug in: $L((x^r)^n) = L(x^m)$ (Why?) $= mL(x)$; on the other hand, $L((x^r)^n) = nL(x^r)$. Set the two expressions equal to each other: $nL(x^r) = mL(x)$; it follows that $L(x^r) = \frac{m}{n}L(x) = rL(x)$, as desired.

Now it is safe to change notation, and define:

$$\ln(x) = L(x), \quad x > 0$$

Since we know that $\ln(x)$ is a one-to-one function that takes on all real values, it is now perfectly correct to *define*:

$$e \text{ is the unique real number with } \ln(e) = 1$$

Exercise 2.6 How do we now know that $\ln(e^r) = r$ for rational numbers r ?

3 $\exp(x)$

We know from the last section that $\ln(x)$ is invertible, so it has an inverse function:

$$\exp(x) \text{ is the inverse function to } \ln(x)$$

In other words, $y = \exp(x)$ if and only if $x = \ln(y)$. The following properties of $\exp(x)$ are now straightforward:

1. $\exp(x)$ has domain \mathbb{R} and range $(0, \infty)$

This is just because the domain of a function is the range of its inverse, and vice versa.

To prove the next two properties, we can just take the \ln of both sides, and use the fact that if $\ln(r) = \ln(s)$ then $r = s$. So, for example, since $\ln(\exp(0)) = 0 = \ln(1)$, $\exp(0) = 1$. Similarly, since $\ln(\exp(1)) = 1 = \ln(e)$ (that's how we *defined* e !), $\exp(1) = e$.

2. $\boxed{\exp(0) = 1 \text{ and } \exp(1) = e}$
3. $\boxed{\exp(a + b) = \exp(a) \exp(b) \text{ for all } a, b}$

Exercise 3.1 *Prove this by taking logarithms of both sides.*

4. $\boxed{\exp(x) \text{ is differentiable, with derivative } \exp(x)}$

Differentiability follows from the theorem on differentiation of inverses of functions in the text. The easiest way to see that $\frac{d}{dx} \exp(x) = \exp(x)$ is to apply implicit differentiation to the equation $\ln(\exp(x)) = x$

Exercise 3.2 *Do it.*

We know that $\ln(e^r) = r$ for rational r , so that $e^r = \exp(r)$ for such r . It is therefore reasonable to extend this formula for e^r to *irrational* r :

$$\boxed{e^r = \exp(r) \text{ for all } r \in \mathbb{R}}$$

I can't emphasize too strongly that the above is the *definition* of e^r for irrational r , which wouldn't otherwise make any sense at all.

The more general exponentials and logarithms can now be defined appropriately:

$$\boxed{a^b = e^{b \ln(a)} \text{ for } a > 0 \text{ and arbitrary } b}$$

$$\boxed{\log_a b = \frac{\ln(b)}{\ln(a)} \text{ for } a, b > 0}$$

Of course, we could have just defined one of the above functions, and obtained the other as its inverse. Instead, we assert the following as a *consequence* of these definitions:

Exercise 3.3 *Prove that $\log_a(x)$ and a^x are inverse functions.*

I'll now leave all the properties of these functions as appear in the text – including formulas for the derivative and integral of a^x and $\log_a(x)$ – to you. Here are a couple, as an exercise:

Exercise 3.4 *Show that $\frac{d}{dx} a^x = a^x \ln(a)$ and $\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$*

(Hint: Just use the above definitions of these functions, and the formulas we've derived for the derivatives of $e^x = \exp(x)$ and $\ln(x)$.)

I'll finish with a formula you've been using for a while, but *without* a correct proof (or even proper definition for the formula):

Theorem 3.1 $\frac{d}{dx}x^r = rx^{r-1}$

PROOF. $\frac{d}{dx}x^r = \frac{d}{dx}e^{r \ln(x)} = e^{r \ln(x)} \frac{d}{dx}(r \ln(x)) = (x^r) \left(\frac{r}{x}\right) = rx^{r-1} \quad \dashv$