

### Examples of differential equations

(or situations which can be modeled with differential equations:

1. The rate of change of  $y$  is constant. (For example, you are driving at a constant speed.)

$$\frac{dy}{dt} = k \quad (k \text{ constant})$$

Sol'n

(i) Separation of vbles:

$$dy = k dt$$

$$\int dy = \int k dt$$

$$\boxed{y = kt + C}$$

OR (ii) We know (or guess) that  $y = kt$  is one sol'n.

if  $y = g(t)$  is any other sol'n, then  $g'(t) = k$ ,

so  $(kt)' - g'(t)$  is constant (namely, 0)

so  $g(t)$  differs from  $kt$  by some constant ( $C$ , say)

$$\therefore g(t) = kt + C$$

2. The rate of change of  $y$  is proportional to  $y$ .

(For example:

- **Population Growth:**  $y(t)$  represents size of population at time  $t$
- **Radioactive Decay:**  $y(t)$  represents the amount of plutonium remaining after  $t$  years.
- **Hazard Rate:** (later if time)  $S(t)$  represents the probability that a hard drive is still working after  $t$  hours.

$$\frac{dy}{dt} = k y$$

(i) Separation:

$$\frac{dy}{y} = k dt$$

$$\int \frac{dy}{y} = \int k dt$$

$$\ln|y| = kt + C$$

$$|y| = e^{kt+C} = e^C e^{kt}$$

$$y = C e^{kt}$$

↖ new constant ( $= \pm e^{\text{old } C}$ )

(ii) Suppose  $y = g(t)$  satisfies  $\frac{dy}{dt} = ky$ ,  $g'(t) = kg(t)$

Know (or guess)  $y = e^{kt}$  is a soln,  $\frac{dy}{dt} = ke^{kt}$

$$g'(t)e^{-kt} = ke^{-kt}g(t) \quad \leftarrow \text{magic? ("integrating factor")}$$

$$g'(t)e^{-kt} - ke^{-kt}g(t) = 0$$

$$= \frac{d}{dt}(g(t)e^{-kt}) !$$

$$\text{so } \frac{d}{dt}(g(t)e^{-kt}) = 0$$

$$\text{so } g(t)e^{-kt} = C, \text{ some constant } C$$

$$g(t) = Ce^{kt}$$

4. Suppose Cooperstown had a population of 25000 in 1970 and 30000 in 1980. Assuming exponential population growth, what population do we project for 2010?

$y(t)$  = population at time  $t$

$t$  = years since 1970

Given:  $y(0) = 25000$   
 $y(10) = 30000$  } initial conditions

Want  $y(40)$

"Exponential growth" means

$$y = ce^{kt}, \text{ some constants } c, k$$

Find  $c, k$  using the initial conditions

$$y(0) = 25000 \Rightarrow ce^{k \cdot 0} = 25000 \Rightarrow c = 25000$$

$$y(10) = 30000 \Rightarrow ce^{k \cdot 10} = 30000 \Rightarrow e^{10k} = \frac{30000}{25000}$$

$$\Rightarrow 10k = \ln(6/5) \Rightarrow k = \frac{1}{10} \ln(6/5)$$

so  $y(t) = 25000 e^{(t/10)(\ln 6/5)}$

$$y(40) = 25000 e^{\frac{4 \ln(6/5)}{10}} = 25000 \left(\frac{6}{5}\right)^4$$

$$= 51840$$