Inverse functions

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1 Review

Some examples of inverse functions you already know:

1. $f(x) = x^2$ and $g(x) = \sqrt{x}$
2. $f(x) = x^3$ and $g(x) = x^{1/3}$
3. $f(x) = e^x$ and $g(x) = \ln x$
4. $f(x) = \sin x$ and $g(x) = \arcsin x$

Note:

- In each case $f(g(x)) = x$ on $domain(g)$
- $g(f(x)) = x$ on $domain(f)$ provided we are careful in specifying $domain(f)$

Write $f^{-1}(x)$ for the inverse function of f (if it exists)

Warning: Not the same thing as $\frac{1}{f(x)}$

EG $f(x) = 3x^3 - 12$; then $f^{-1}(x) = \sqrt[3]{(x+12)/3}$ (Show)

This works because we can solve for y in terms of x. How do we make this idea mathematically precise?

Definition 1.1 *A function f is* one-to-one *provided two different x-values do not map onto the same y-value; that is,*

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

For example, a strictly monotonic (=strictly increasing or strictly decreasing) function is one-to-one.
Remark: If f is strictly monotonic, then so is f^{-1}, with the same sense.

Proposition 1.1 *f has an inverse (=is invertible) if and only if it is one-to-one.*

Remark: We often identify a function f with its graph $\{\langle x, y\rangle : y = f(x)\}$; then $f^{-1} = \{\langle x, y\rangle : x = f(y)\}$ (picture)

2 Calculus

Theorem 2.1 *(a) If f is an invertible function which is continuous on an interval I then f is strictly monotonic on the interval, and f^{-1} is also continuous.*

(b) If f is an invertible function which is differentiable on an open interval I then f^{-1} is also differentiable; moreover, $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f(x))}$

Proof 2.1 *Class*