Root Test and

Ratio Test (for a nonnegative sequence $\{a_n\}_{n}$)

ⓐ Ratio test:

If $\frac{a_{n+1}}{a_n} \to \lambda$ as $n \to \infty$, then

① if $\lambda < 1$, $\sum a_n$ converges
② if $\lambda > 1$, $\sum a_n$ diverges
③ if $\lambda = 1$, No Conclusion

ⓑ Root test:

If $\sqrt[n]{a_n} \to \rho$ as $n \to \infty$, Then

① if $\rho < 1$, $\sum a_n$ converges
② if $\rho > 1$, $\sum a_n$ diverges
③ if $\rho = 1$, No Conclusion

EG $\sum_n ar^n$ (Geometric Series) ; $a_n = ar^n$

ratio: $\frac{a_{n+1}}{a_n} = \frac{ar^{n+1}}{ar^n} = r \to r$ as $n \to \infty$

if $r<1$, converges; if $r>1$, diverges (we knew this!)

if $r=1$, no conclusion (we know series diverges, but test says nothing)

root $a_n^{1/n} = (\sqrt[n]{a})\, r \to r$ as $n \to \infty$

$\therefore$ Same conclusions as above

EG $\sum \frac{n^2}{n!}$ $\quad a_n = \frac{n^2}{n!}$

root test $\sqrt[n]{a_n} = \left(n^{\frac{1}{n}}\right)^2 / \sqrt[n]{n!}$ - can't compute limit!

ratio test $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(n+1)!} \Big/ \left(\frac{n^2}{n!}\right) = \underbrace{\left(\frac{n!}{(n+1)!}\right)}_{=\frac{1}{n+1}} \left(\frac{n+1}{n}\right)^2 = \frac{n+1}{n^2} \to 0$ as $n \to \infty$

Since $0<1$, series converges

REMARK Since $\sum \frac{n^2}{n!}$ converges, we get 'for free' that $\frac{n^2}{n!} \to 0$ as $n \to \infty$

EG $\sum \frac{n}{2^n}$ $\quad a_n = \frac{n}{2^n}$

root: $\sqrt[n]{a_n} = \frac{\sqrt[n]{n}}{\sqrt[n]{2^n}} = \frac{\sqrt[n]{n}}{2} \to \frac{1}{2}$ as $n \to \infty$. $\frac{1}{2} < 1$, so

The series $\sum \frac{n}{2^n}$ converges by root test

ratio: $\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{2^{n+1}}\right) \Big/ \left(\frac{n}{2^n}\right) = \left(\frac{2^n}{2^{n+1}}\right)\left(\frac{n+1}{n}\right) = \frac{1}{2}\left(1+\frac{1}{n}\right) \to \frac{1}{2}$ as $n \to \infty$

Since $\frac{1}{2}<1$, series converges by ratio test

EG $\sum \underbrace{\frac{n!}{n^n}}_{a_n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}}$$

$$= (n+1)\left(\frac{n^n}{(n+1)^{n+1}}\right) = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} \text{ as } n \to \infty$$

$e \simeq 2.17\ldots$, so $\frac{1}{e} < 1$, so $\sum \frac{n!}{n^n}$ converges.

by ratio test

EG $\sum \frac{1}{n}$:

root: $\sqrt[n]{a_n} = \left(\frac{1}{n}\right)^{1/n} = \frac{1}{\sqrt[n]{n}} \to 1$ as $n \to \infty$, no information

ratio: $\frac{a_{n+1}}{a_n} = \left(\frac{1}{n+1}\right) / \left(\frac{1}{n}\right) = \frac{n}{n+1} \to 1$ as $n \to \infty$, no information

WARNING note $\frac{1}{\sqrt[n]{n}} < 1$, also $\frac{n}{n+1} < 1$; only limit is important

EG $\sum \frac{1}{n^2}$:

root $\sqrt[n]{a_n} = \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \left(\frac{1}{n^{1/n}}\right)^2 = \left(\frac{1}{\sqrt[n]{n}}\right)^2 \to \frac{1}{1^2} = 1$ as $n \to \infty$

ratio $\frac{a_{n+1}}{a_n} = \left(\frac{1}{(n+1)^2}\right) / \left(\frac{1}{n^2}\right) = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 \to 1$ as $n \to \infty$

So again: Neither test gives information

Series with Negative Terms

EG We know $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges

What about $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$?

"Alternating harmonic series"

Rewrite: $$\text{“}=\text{”} \sum_{n=0}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(2n+2)-(2n+1)}{(2n+1)(2n+2)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4n^2+6n+2} \quad \text{Converges!}$$

DEF If $\sum |a_n|$ converges, say $\sum a_n$ converges absolutely

If $\sum |a_n|$ diverges, but $\sum a_n$ converges,

say $\sum a_n$ converges conditionally

THEOREM If $\sum |a_n|$ converges then $\sum a_n$ converges

PRF $a_n = (a_n + |a_n|) - |a_n|$, so suffices to show $\sum (a_n + |a_n|)$ converges

but: $0 \leq a_n + |a_n| \leq 2|a_n|$, and $\sum 2|a_n|$ converges

so $\sum (a_n + |a_n|)$ converges by comparison test.

DEF $\sum a_n$ is an <u>alternating</u> <u>series</u> if the signs alternate - That is, $\text{sign}(a_n) = -\text{sign}(a_{n+1})$ for all n

EG $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ is alternating

$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots$ is <u>not</u> alternating

THM ("Alternating Series test") If $\sum a_n$ is an alternating series and $|a_n|$ monotonically decreases to 0 then $\sum a_n$ converges ("bad test becomes good test")

EG $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n^2+1} = 0 - \frac{1}{2} + \frac{2}{5} - \frac{3}{10} + \frac{4}{17} - \cdots$

alternating ✓

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$ ✓

Monotonic: ① If $f(x) = \frac{x}{x^2+1}$, $f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ on $[1,\infty)$, so ↓

-OR- ② $\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{(n+1)}{(n+1)^2+1}\right) \Big/ \left(\frac{n}{n^2+1}\right) = \frac{(n+1)(n^2+1)}{(n^2+2n+2)\,n} = \frac{n^3+n^2+n+1}{n^3+2n^2+2n} < 1$

so $|a_{n+1}| < |a_n|$

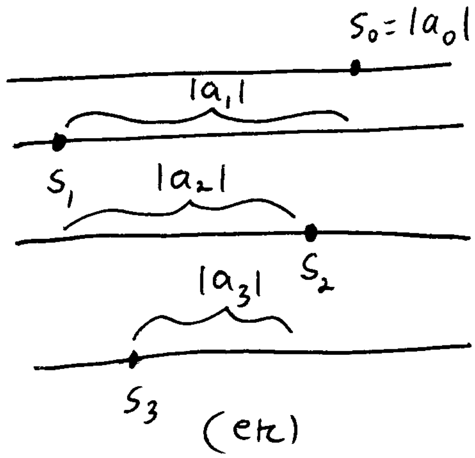
-OR- ③ (Exercise) show $|a_n| - |a_{n+1}| > 0$

<u>EG</u> $\sum(-1)^n \frac{n}{n+1}$ Since $a_n \not\to 0$, series <u>diverges</u>

<u>Proof of alternating series test:</u>

Suppose (for definiteness) series starts with a positive term, for example $\sum_{n=0}^{\infty}(-1)^n|a_n|$

Plot successive partial sums:



$S_0 \geq S_2 \geq S_4 \geq S_6 \geq \cdots \geq S_5 \geq S_3 \geq S_1$

and $|S_{2n+1}-S_{2n}| = |a_{2n+1}| \to 0$ as $n\to\infty$

(Why is this enough?)

<u>COROLLARY</u> (IMPORTANT) If $\sum a_n$ an alternating series and $\sum a_n = L$ then $|S_N - L| < |a_{n+1}|$

<u>EG</u> To find $(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots)$ within $\frac{1}{10000}$, suffices to take $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{9999}$