Sequences

A **sequence** is a function $f: \mathbb{N} \to \mathbb{R}$

Intuitively, the sequence is the 'list' $f(0), f(1), f(2), f(3), \dots, f(n), \dots$

Notation: If $a_n = n^{th}$ term of sequence $= f(n)$, write
$\{a_n\}_n$ or just a_n to denote the sequence

Def $\{a_n\}_n$ is **increasing** if for all $m<n$, $a_m < a_n$
" " **nondecreasing** " " $m<n$, $a_m \le a_n$
" " **decreasing** " " " $a_m > a_n$
" " **nonincreasing** " " " $a_m \ge a_n$

Def $\{a_n\}_n$ is a **monotonic** sequence if it is either
nondecreasing or nonincreasing

$\{a_n\}_n$ is **bounded above** if for some M, $a_n \le M$ for all n
" " " **below** " " " N, $N \le a_n$ for all n

$\{a_n\}_n$ is **bounded** if it is bounded above <u>and</u> below.

Limits

Intuitive $\lim_{n\to\infty} a_n = L$ means: as n gets infinitely large, a_n gets infinitely close to L

Rigorous $\lim_{n\to\infty} a_n = L$ provided: For any $\varepsilon > 0$, there is a N large enough that: for any $n > N$, $|a_n - L| < \varepsilon$.

Also $\lim_{n\to\infty} a_n = \infty$ if: (intuitively) as n gets infinitely large, a_n also gets arbitrarily large.

(Similarly for $\lim_{n\to\infty} a_n = -\infty$)

EG $\lim_{n\to\infty} \frac{n^2+n}{3n^2-2n} = \lim_{n\to\infty} \frac{1+1/n}{3+2/n} = \frac{1}{3}$

EG $\lim_{n\to\infty} \frac{n^2+1}{3n+2} = \infty$

Def $\{a_n\}_n$ **converges** if $\lim_{n\to\infty} a_n = L$ for some (finite) $L \in \mathbb{R}$

otherwise $\{a_n\}_n$ **diverges**

If $\lim_{n\to\infty} a_n = \infty$, say $\{a_n\}_n$ **diverges to infinity**

Properties of limits (Just like Calc I)

(1) If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} a_n = M$ then $L = M$

(2) If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ then

$$\lim_{n\to\infty}(a_n+b_n) = L+M, \quad \lim_{n\to\infty}(a_n b_n) = LM, \quad \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M} \quad (\text{if } M \neq 0)$$

(3) $\lim\limits_{n\to\infty} a_n = L \Leftrightarrow \lim\limits_{n\to\infty} |a_n - L| = 0 \Leftrightarrow \lim\limits_{n\to\infty} (a_n - L) = 0$

(4) (Pinching/squeezing Theorem) If $a_n \leq b_n \leq c_n$ and $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$

then $\lim\limits_{n\to\infty} b_n = L$

Eg $\lim\limits_{n\to\infty} n^2 \sin(\frac{1}{n})/(n+1) = \lim\limits_{n\to\infty} \overset{1}{\left(\frac{n^{\cancel{2}}}{n+1}\right)} \overset{1}{\left(\frac{\sin(1/n)}{1/n}\right)} = 1 \cdot 1 = 1$

Eg $\lim\limits_{n\to\infty} \frac{\sin n + \cos n}{n} = ?$ (class)

More limit properties

Prop. If $a_n = f(n)$ for some ~~$f:\mathbb{R}\to\mathbb{R}$~~ and $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} a_n = L$

Eg $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = ?$ $\quad \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \underset{\text{L'Hospital}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0,$ ✱

so $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

Prop. If $f:\mathbb{R}\to\mathbb{R}$ is continuous at x_0 and $\lim\limits_{n\to\infty} a_n = x_0$ then $\lim\limits_{n\to\infty} f(a_n) = f(x_0)$

Eg Let $a_n = n^{1/n}$ $(n=1,2,3,\dots)$ i.e., $1, \sqrt{2}, \sqrt[3]{3}, \dots, \sqrt[n]{n}, \dots$

Then $\ln(a_n) = \frac{1}{n}\ln n = \dfrac{\ln n}{n} \to 0$ as $n\to\infty$ ✱

So $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} e^{\ln(a_n)} = e^0 = 1$

TWO OTHER OCCASIONALLY USEFUL FACTS:

Thm (1) If $\lim\limits_{n\to\infty} a_n$ exists then $\{a_n\}_n$ is bounded

(2) If $\{a_n\}_n$ is bounded and monotonic then $\lim\limits_{n\to\infty} a_n$ exists.

✱ = Important examples!

A useful hierarchy

Each of the following $\to \infty$ faster than the ones below it (in sense that ratio of lower to higher $\to 0$)

$n! = n(n-1)(n-2)\cdots(3)(2)(1)$

e^n (or a^n, any $a>1$)

n^p ($p>1$)

n

$(\ln n)^k$ ($k>1$)

$\ln n$

Some we already know:

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\lim_{n\to\infty} \frac{n}{n^p} = \lim_{n\to\infty} \frac{1}{n^{p-1}} = 0$$

etc

- will do some others in class on blackboard

2 more useful examples:

EG If $x > 0$, $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$

Proof: Take logarithms (exercise! Do it!); or:

For large enough n, $\frac{1}{n} \leq x \leq n$

so $$\frac{1}{n^{1/n}} = \left(\frac{1}{n}\right)^{1/n} \leq x^{1/n} \leq n^{\frac{1}{n}}$$

$n^{1/n} \to 1$, $n^{\frac{1}{n}} \to 1$

$\therefore \lim_{n\to\infty} x^{\frac{1}{n}} = 1$ by squeezing theorem.

EG $\lim_{n\to\infty} (1+\frac{x}{n})^n = e^x$:

$a_n = (1+\frac{x}{n})^n$, so $\ln a_n = n \ln(1+\frac{x}{n}) = \dfrac{\ln(1+\frac{x}{n})}{(1/n)}$

$$= \frac{\ln(1+\frac{x}{n}) - \ln(1)}{(x/n)} \cdot x$$

$\to x\, g'(1)$ where $g(x) = \ln(x)$

$= x$

so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln(a_n)} = e^x$