# Series

**Recall**

$$\sum_{k=m}^{n} a_k := a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

**Define** $\displaystyle\sum_{k=m}^{\infty} a_k = \lim_{N\to\infty} \sum_{k=m}^{N} a_n$ (if the limit exists

& is finite, in which case we say the sum converges;

if $\displaystyle\lim_{N\to\infty} \sum_{k=m}^{N} a_k$ does not exist, say $\displaystyle\sum_{k=m}^{\infty} a_k$ diverges.

**Idea:** start with one sequence $a_1, a_2, a_3, a_4, \cdots$

Form partial sums:

$$\underbrace{\underbrace{\underbrace{a_1 + a_2}_{s_2} + a_3}_{s_3} + a_4}_{s_4 \ldots} + a_5$$

$s_N$ = "nth partial sum"

$$= a_1 + \cdots + a_N$$
$$= \sum_{n=1}^{N} a_n$$

Note $\{s_n\}_{n=1}^{\infty}$ is another sequence, different from $\{a_n\}_{n=1}^{\infty}$

We say the series $\displaystyle\sum_{n=1}^{\infty} a_n = L$ provided

the sequence $s_N \to L$

**EG** $\sum_{n=1}^{\infty} c$ , $c$ constant $\neq 0$, say $>0$

$$S_N = \overbrace{c+c+c+\cdots+c}^{N} = Nc \to \infty \text{ as } N\to\infty \quad \therefore \sum_{n=1}^{\infty} c \text{ diverges}$$

**EG** (Geometric Series) $\sum_{n=0}^{\infty} ar^n \left(= a+ar+ar^2+\cdots\right)$

$$S_N = a+ar+ar^2+\cdots+ar^N$$
$$rS_N = ar+ar^2+ar^3+\cdots+ar^{N+1}$$

$$(1-r)S_N = a - ar^{N+1} \qquad \text{(subtract)}$$

$$\therefore S_N = \frac{a-ar^{N+1}}{1-r} = a\left(\frac{1-r^{N+1}}{1-r}\right)$$

If $|r|<1$, this $\to \frac{a}{1-r}$ as $N\to\infty$

if $|r|>1$, this $\to \pm\infty$, diverges

If $|r|=1$, above algebra fails.

if $r=1$: $\sum_{n=0}^{\infty} a$ , diverges if $a\neq 0$

if $r=-1$: $\sum_{n=0}^{\infty} a(-1)^n = a-a+a-a+a-a+\cdots$

partial sums: $a$, $0$, $a$, $0$, …

so $\lim_{N\to\infty} S_N$ does not exist

$\therefore$ diverges.

**So**: $$\boxed{\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r}, & |r|<1 \\ \text{diverges otherwise} \end{cases}}$$

EG $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=?$

geometric series, $a = 1^{st}$ term $= 1/2$, $r = 1/2$

$\therefore$ sum $= \frac{1/2}{1-1/2} = 1$

EG $0.99999\cdots = \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots$ geometric, $a = 9/10$, $r = 1/10$

$\therefore = \frac{(9/10)}{1-(1/10)} = \frac{9}{10-1} = 1$, so $0.99999\cdots = 1$ (!)

EG $\frac{1}{4-x^2} = \frac{1/4}{1-x^2/4} = \frac{1/4}{1-(x/2)^2}$ $\boxed{\begin{array}{l} a = 1/4 \\ r = x^2/4 \end{array}}$ $= \sum_{n=0}^{\infty}(1/4)\left(\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\frac{x^{2n}}{4^{n+1}}$

(as long as $\frac{x^2}{4}<1$, i.e., $|x|<2$)

EG (Harmonic Series) $\sum_{n=1}^{\infty}\frac{1}{n} = ?$

$$1+\underbrace{\tfrac{1}{2}}_{=1/2}+\underbrace{\tfrac{1}{3}+\tfrac{1}{4}}_{\geq 1/2}+\underbrace{\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}}_{\geq 1/2}+\underbrace{\tfrac{1}{9}+\cdots+\tfrac{1}{16}}_{\geq 1/2}+\cdots$$

so $S_{2^N} \geq 1+\frac{N}{2} \to \infty$ as $N\to\infty$, so $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

(WARNING:

NOTE $\sum_{n=1}^{\infty}\frac{1}{n} = \infty$ but $\lim_{n\to\infty}\frac{1}{n} = 0$ ! )

EG $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2} = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \cdots$ ??

$$\frac{2n+1}{(n^2+n)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}$$

$$\begin{array}{c} // \\ n^2(n+1)^2 \end{array}$$

$$2n+1 = A(n)(n+1)^2 + B(n+1)^2 + C(\cancel{n^2}(n+1) + D(n^2)$$

$\underline{n=0}$: $1 = B$ $\qquad\qquad$ $\underline{n^3}$: $0 = A + C$

$\underline{n=-1}$: $-1 = D$ $\qquad\qquad$ $\underline{n}$: $2 = A + 2B \} \therefore A = 0 \quad \therefore C = 0$

$\underline{\text{so}}$: $\displaystyle\frac{2n+1}{(n^2+n)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$, so $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) - \left(\frac{1}{(n+1)^2}\right)$

$S_N = N^{th}$ partial sum $= \left(1 - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{16}}\right) \cancel{+} \cdots + \left(\cancel{\frac{1}{N^2}} - \frac{1}{(N+1)^2}\right)$

$$= 1 - \frac{1}{(N+1)^2} \rightarrow 1 \text{ as } N \rightarrow \infty$$

$$\therefore \sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2} = 1$$

This is called a <u>telescoping sum</u>

Form: $\displaystyle\sum_{n=1}^{\infty} (b_n - b_{n+1})$ ; if $b_n \rightarrow 0$ as $n \rightarrow \infty$,

This converges to $b_1$

# Properties of series

**Theorem** Suppose $\sum_{n=k}^{\infty} a_n = L$, $\sum_{n=k}^{\infty} b_n = M$, and $c$ a constant

(1) $\sum_{n=k}^{\infty} (a_n + b_n) = L + M$ (2) $\sum_{n=k}^{\infty} c a_n = cL$

(3) $\lim_{n\to\infty} a_n = 0$

**Proof** (Discuss in class / board)

**Corollary** If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges

(Why is this a corollary?)

**EG** $\sum_{n=1}^{\infty}\left(\frac{17}{5\cdot 2^n} + \left(-\frac{5}{8}\right)^{3n}\right) = ?$

$\sum_{n=1}^{\infty} \frac{17}{5\cdot 2^n} = \left(\frac{17}{5\cdot 2}\right)/\left(1-\frac{1}{2}\right) = 17/5$

$\sum_{n=1}^{\infty} \left(-\frac{5}{8}\right)^{3n} = \sum_{n=1}^{\infty}\left[\left(-\frac{5}{8}\right)^3\right]^n = \frac{(-5/8)^3}{1-(-5/8)^3} = \frac{-125}{8^3+125}$

both converge, so $\sum_{n=1}^{\infty}\left(\frac{17}{5\cdot 2^n} + \left(-\frac{5}{8}\right)^{3n}\right) = \frac{17}{5} - \frac{125}{512+125}$

**EG** $\sum_{n=1}^{\infty} \frac{n^2+1}{7n^2+n+1} = ?$

Since $\lim_{n\to\infty} \frac{n^2+1}{7n^2+n+1} = \lim_{n\to\infty} \frac{1+1/n^2}{7+1/n+1/n^2} = 1/7 \neq 0$,

The series diverges!

# Some Warnings

(1) For a <u>sequence</u> $a_n$, both the convergence of $a_n$ and the limit of $a_n$ do not depend on the initial terms: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+5}$ (or both diverge)

For a <u>series</u> $\sum a_n$, convergence does not depend on initial terms, but the actual sum does:

$$\sum_{n=1}^{\infty} a_n \text{ converges if \& only if } \sum_{n=k}^{\infty} a_n \text{ converges } (k>1)$$

but $\sum_{n=1}^{\infty} a_n \neq \sum_{n=5}^{\infty} a_n$

(2) If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$ ; but:

If $\lim_{n\to\infty} a_n = 0$ you <u>can't conclude</u> that $\sum a_n$ converges.

(3) Even if $\sum a_n$ and $\sum b_n$ converge, usually $(\sum a_n)(\sum b_n) \neq \sum (a_n b_n)$ (Exercise - find a counterexample)

(4) DON'T TRUST COMPUTERS -

If $a_n \to 0$ then $\sum a_n$ will seem to converge on your computer!