Cardinal Exercises

1. Let κ be an infinite cardinal, and κ^+ its cardinal successor. Show that $\kappa^+ \smallsetminus \kappa$ (that is, $\{\alpha | \kappa \leq \alpha < \kappa^+\}$) has cardinality κ^+

2. If X is an infinite set, show that $\mathcal{P}_f(X) = \{a \subset X | a \text{ is finite}\}$ has the same cardinality as X.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is called *upper semicontinuous* (USC) provided there is a sequence of continuous functions $\{f_n\}_{n \in \omega}$ such that for all x $f_0(x) \geq f_1(x) \geq f_2(x) \geq \cdots$, and $f(x) = \lim_{n \to \infty} f_n(x)$. What is the cardinality of the set of all USC functions? (You might recall that there are only $2^{\aleph_0}$ many continuous functions).

4. Let A and B be fixed sets. Show that there is a one-to-one function f with $dom(f) \subseteq A$ and $range(f) \subseteq B$ and such that f cannot be extended to a one-to-one function f' with $dom(f) \subsetneq dom(f') \subseteq A$ and $range(f') \subseteq B$. Extra points for doing this two ways (ie, using two different versions of AC). (Something to think about: is this problem *equivalent* to AC?)