Exercises 8(?)

1. Let $\mathcal{L} = \{P\}$ be the language with precisely one unary predicate symbol P. Suppose $A = \{1, 2, 3, \cdots, N\}$ for some natural number N. How many different $\mathcal{L}$ -structures are there with domain A?

2. Let $\mathcal{L} = \{P_1, P_2\}$ be the language with two unary predicate symbols P_1 and P_2. Suppose $A = \{1, 2, 3, \cdots, N\}$ for some natural number N. How many different $\mathcal{L}$ -structures $\mathfrak{A}$ with domain A are there such that $\mathfrak{A} \models \forall x(P_1(x) \Rightarrow P_2(x))$?

3. Let $\mathcal{L}$ be a first-order language, and let $\mathfrak{A} = \langle A, \cdots \rangle$ and $\mathfrak{B} = \langle B, \cdots \rangle$ be two $\mathcal{L}$ -structures. Recall that a function $\phi : A \to B$ is a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ provided it "respects" $\mathcal{L}$; in other words,

   1. $\phi(c_{\mathfrak{A}}) = c_{\mathfrak{B}}$ for every constant symbol c;
   2. If f is an n-ary function symbol, and $a_1, \cdots, a_n \in A$ then $\phi(f_{\mathfrak{A}}(a_1, \cdots, a_n)) = f_{\mathfrak{B}}(\phi(a_1), \cdots, \phi(a_n))$; and
   3. If P is an n-ary predicate symbol, and $a_1, \cdots, a_n \in A$ then $P_{\mathfrak{A}}(a_1, \cdots, a_n)$ holds (in $\mathfrak{A}$) if and only if $P_{\mathfrak{B}}(\phi(a_1), \cdots, \phi(a_n))$ holds (in $\mathfrak{B}$).

   (a) Show that $\phi(x) = 1 - x$ is a homomorphism from $\langle \mathbb{N}, < \rangle$ to $\langle \mathbb{Q}, > \rangle$. (Note that this function is one-to-one; one-to-one homomorphisms are sometimes called *isomorphisms*.)

   (b) Find an isomorphism from $\langle \mathbb{R}, 0, + \rangle$ to $\langle \mathbb{R}^+, 1, \times \rangle$. (Here R^+ is the set of positive reals.) (Hint: the answer is $\phi(x) = e^x$; show that it works!)

4. Let $\mathcal{L}$ be a language with one binary relation symbol E. Let $\mathfrak{A}$ be the $\mathcal{L}$ -structure with domain $A = \{(a, b) \in \mathbb{N}^2 | b \neq 0\}$ and where $E_{\mathfrak{A}} = \{((a, b), (c, d)) \in A \mid ad = bc\}$. Show that $E_{\mathfrak{A}}$ is an equivalence relation on A.