Sol'ns from Handout with problems # ① – ⑧

① (#4) HINT: 3 eq'ns in 3 unknowns a, b, c. 1^{st} eq'n: f is a pdf, so $\int_0^1 f(x)\,dx = 1$

2^{nd} eq'n: $\frac{1}{3} = E(X) = \int_0^1 x f(x)\,dx$ $\quad 3^{rd}$: $Var(X) = \frac{1}{18} = E(X^2) - E(X)^2 = \int_0^1 x^2 f(x)\,dx - \frac{1}{9}$.

② (#8) $Y = \cos X$, $Z = \sin X$, $X \sim \mathcal{U}(0, 2\pi)$, $E(YZ) = \frac{1}{2\pi}\int_0^{2\pi} \sin x \cos x\,dx$

$= \frac{1}{4\pi}\int_0^{2\pi} \sin 2x\,dx = 0$, $E(Y) = \frac{1}{2\pi}\int_0^{2\pi} \cos x\,dx = 0 = \frac{1}{2\pi}\int_0^{2\pi} \sin x\,dx = E(Z)$.

$\therefore Cov(Y, Z) = E(XY) - EXEY = 0$

(so Y & Z are uncorrelated, but not independent (show!))

③ (#3) If $X = i$ then Urn II contains $2+i$ white balls and $1 + (2-i) = 3-i$ black ones after the 1st transfer, so

$$f(y|i) = \begin{cases} \binom{3-i}{2} \Big/ \binom{5}{2} & y = 0 \\ (2+i)(3-i) \Big/ \binom{5}{2} & y = 1 \\ \binom{2+i}{2} \Big/ \binom{5}{2} & y = 2 \end{cases} = \frac{1}{30}\begin{cases} \binom{3-i}{2} & y = 0 \\ (2+i)(3-i) & y = 1 \\ \frac{(2+i)(1+i)}{2} & y = 2 \end{cases}$$

$$\therefore E(Y|X=i) = 0 f(0|X=i) + 1 f(1|i) + 2 f(2|i) = \frac{1}{30}\left[(2+i)(3-i) + (2)\left(\tfrac{1}{2}\right)(2+i)(1+i)\right]$$

$$= \frac{1}{30}(2+i)(3-i+1+i) = \frac{2}{15}(2+i), \quad i = 0, 1, 2$$

⑧ (#5) $f(x,y) = 2 - x - y\ \mathbb{1}_{0<x<1}\mathbb{1}_{0<y<1}$ $\quad \therefore f_X(x) = \int_0^1 (2-x-y)\,dy = 2y - xy - \frac{y^2}{2}\Big|_0^1 = (2-x) - \frac{1}{2}, \quad 0<x<1$

$$\therefore f(y|x) = \frac{f(x,y)}{f(x)} = \frac{2-x-y}{\frac{3}{2}-x}\mathbb{1}_{0<y<1}. \quad \therefore E(Y|x) = \int y f(y|x)\,dy = \int_0^1 \left(\frac{y}{\frac{3}{2}-x}\right)(2-x-y)\,dy$$

$$= \frac{\int_0^1 2y - xy - y^2\,dy}{\frac{3}{2}-x} = \frac{\frac{y^2}{2}(2-x) - \frac{y^3}{3}\Big|_0^1}{\frac{3}{2}-x} = \frac{\frac{2-x}{2} - \frac{1}{3}}{\frac{3}{2}-x} = \frac{6-3x-2}{9-6x}$$

$$= \frac{4-3x}{9-6x}, \quad 0<x<1$$

$$E(Y^2|x) = \int y^2 f(y|x)\,dx = \frac{1}{(\frac{3}{2}-x)}\int_0^1 (2-x)y^2 - y^3\,dy = \frac{\frac{(2-x)}{3} - \frac{1}{4}}{\frac{3}{2}-x} = \frac{8-4x-3}{18-12x} = \frac{5-4x}{18-12x}$$

$$\therefore Var(Y|x) = E(Y^2|x) - E(Y|x)^2 = \left(\frac{5-4x}{18-12x}\right) - \left(\frac{4-3x}{9-6x}\right)^2$$

Problems from "Section 5.5" handout

(5) $X \sim Poiss(\lambda)$. $P(X \text{ even}) = \sum_{k=0}^{\infty} P(X=2k) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{2k}}{(2k)!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)^{2k}}{(2k)!}$

We either recognize this series as a hyperbolic trig function, or reason as follows; $e^{\lambda} = \sum_k \frac{\lambda^k}{k!}$, and $e^{-\lambda} = \sum_k \frac{(-\lambda)^k}{k!} = \sum_k \frac{(-1)^k \lambda^k}{k!}$, so

$$e^{\lambda} + e^{-\lambda} = \sum_{k} \frac{\lambda^k (1+(-1)^k)}{k!} = 2 \sum_{k \text{ even}} \frac{\lambda^k}{k!} = 2 \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!},$$

so $P(X \text{ even}) = (e^{-\lambda}) \left(\frac{e^{\lambda} + e^{-\lambda}}{2} \right) = \frac{1 + e^{-2\lambda}}{2}$.

(i) $X \sim Poiss(\lambda)$, $\lambda = 2$. (a) If $Y = \#$ orders in 30 minutes, $Y \sim Poiss(3\lambda) = Poiss(6)$, so $P(Y=8) = e^{-6} 6^8/8!$

(b) $P(\text{no orders in 20 minutes}) = P(\text{a } Poiss(4) = 0) = \frac{e^{-4} 4^0}{0!} = e^{-4}$

(c) $E(Poiss(4)) = 4$

Chapter 10 #49, 50

49 $E(X|N=n) = E(X I_{N=n})/P(N=n) = E((Y_1+Y_2+\cdots+Y_n) I_{N=n})/P(N=n)$

(Since if $N(\omega)=n$, then $X(\omega)=Y_1(\omega)+\cdots+Y_n(\omega)$)

$= \sum_{i=1}^{n} E(Y_i I_{N=n})/P(N=n) = \sum_{i=1}^{n} E(Y_i)E(I_{N=n})/P(N=n)$ (by Y_i, N indep)

$= \sum_{i=1}^{n} E(Y_i)$ (since $E(I_{N=n}) = P(N=n)$) $= mn$ since $E(Y_i)=m$.

$\therefore E(X|N)(\omega) = m N(\omega)$, so $E(X) = E(E(X|N)) = E(mN) = mE(N) = m\nu$ ✓

50 Let X = # mistakes on all 3 pages $\sim \text{Poiss}(3\lambda)$, i.e., $f(x|\lambda) = e^{-3\lambda}(3\lambda)^x/x!$, $x \in \mathbb{N}$

Then $E(X|\Lambda=\lambda) = 3\lambda$, so $E(X|\Lambda) = 3\Lambda$.

Then $E(X) = E(E(X|\Lambda)) = E(3\Lambda) = 3E(\Lambda) = \frac{1+2+3+4}{4} \cdot 3 = \frac{30}{4} = \frac{15}{2}$