6.2 / 1, 9

1) $X_i \sim N(\theta, \sigma^2)$, $f_\theta(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\theta)^2/2\sigma^2}$, $L_\theta = \text{const} - \frac{1}{2\sigma^2}(x-\theta)^2$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2\sigma^2}\left(2(x-\theta)(-1)\right) = \frac{1}{\sigma^2}(x-\theta) \qquad \frac{\partial^2 L}{\partial \theta^2} = \frac{-1}{\sigma^2}$$

$\therefore I(\theta) = E\left(\frac{1}{\sigma^2}\right) = \frac{1}{\sigma^2}$, so $\frac{1}{nI(\theta)} = \frac{\sigma^2}{n} = Var(\bar{x})$, so $\bar{x}$ efficient.

9) $$E(X_i) = \int_0^\infty \frac{x\, 3\theta^3}{(x+\theta)^4}\,dx = 3\theta^3 \int_0^\infty \frac{x\,dx}{(x+\theta)^4} = 3\theta^3 \int_0^\infty \left(\frac{x+\theta}{(x+\theta)^4} - \frac{\theta}{(x+\theta)^4}\right) dx$$

$$= 3\theta^3 \left[-\tfrac{1}{2}(x+\theta)^{-2}\Big|_0^\infty + \frac{\theta}{3}(x+\theta)^{-3}\Big|_0^\infty \right] = 3\theta^3\left[\tfrac{1}{2}\left(\frac{1}{\theta^2}\right) - \frac{\theta}{3}\left(\frac{1}{\theta^3}\right)\right]$$

$$= \tfrac{3}{2}\theta - \theta = \frac{\theta}{2} \quad \therefore E(\bar{x}) = \frac{\theta}{2} \therefore E(2\bar{x}) = \theta \text{ unbiased.}$$

$I(\theta)$: $L_\theta = \ln(3\theta^3) - 4\ln(x+\theta) = \ln 3 + 3\ln\theta - 4\ln(x+\theta)$

$$\frac{\partial L_\theta}{\partial \theta} = \frac{3}{\theta} - \frac{4}{x+\theta} \qquad \frac{\partial^2 L_\theta}{\partial \theta^2} = -\frac{3}{\theta^2} + \frac{4}{(x+\theta)^2}$$

$$I(\theta) = E\left(\frac{3}{\theta^2} - \frac{4}{(x+\theta)^2}\right) = \frac{3}{\theta^2} - 4E\left(\left(\frac{1}{x+\theta}\right)^2\right).$$

$$E\left(\left(\frac{1}{x+\theta}\right)^2\right) = 3\theta^3 \int_0^\infty \frac{1}{(x+\theta)^6}\,dx = (3\theta^3)\left(-\frac{1}{5}\right)\left(\frac{1}{x+\theta}\right)^5\Big|_0^\infty = \frac{3\theta^3}{5}\left(\frac{1}{\theta^5}\right) = \frac{3}{5\theta^2}$$

$$\therefore I(\theta) = \frac{3}{\theta^2} - \frac{12}{5\theta^2} = \frac{1}{\theta^2}\left(3 - \frac{12}{5}\right) = \frac{3}{5\theta^2}$$

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$$Var(X_i) = E(X_i^2) - \left(\frac{\theta}{2}\right)^2 = \theta^2 - \left(\frac{\theta}{2}\right)^2 = \frac{3\theta^2}{4}$$

so $Var(2\bar{x}) = 4 \cdot \left(\frac{3\theta^2}{4}\right)/n = \frac{3\theta^2}{n}$ $\therefore$ Efficiency $= \left(\frac{1}{nI(\theta)}\right) \Big/ \left(\frac{3\theta^2}{n}\right) = \left(\frac{5\theta^2}{3}\right)\frac{1}{3\theta^2}$

$= 5/9$