Problem 1 (§1.4, #20). *Let $\mathcal{C}$ be a collection of sets and E an element in the σ-algebra generated by $\mathcal{C}$. Then there is a countable subcollection $\mathcal{C}_o \subset \mathcal{C}$ such that E is an element of the σ-algebra $\mathcal{A}_o$ generated by $\mathcal{C}_o$.*

Proof. Let $\mathcal{Y}$ be the family of all σ-algebras generated by countable subsets of $\mathcal{C}$, and let $\mathcal{A}' = \bigcup \mathcal{Y}$. We must show that $\mathcal{A}'$ is a σ-algebra.

(i) Since $\varnothing \in \mathcal{B}$ for all σ-algebras $\mathcal{B} \in \mathcal{Y}$, it follows that $\varnothing \in \mathcal{A}'$. (ii) If $A \in \mathcal{A}'$, then $A \in \mathcal{B}_A$ for some σ-algebra $\mathcal{B}_A \in \mathcal{Y}$. Now because $\mathcal{B}_A$ is a σ-algebra, we must have $A^c \in \mathcal{B}_A \subset \bigcup \mathcal{Y} = \mathcal{A}'$. (iii) If $(A_i)_{i=1}^{\infty}$ is a sequence of sets in $\mathcal{A}'$, then for each $i \in \mathbb{N}$, $A_i \in \mathcal{B}_i$ for some σ-algebra $\mathcal{B}_i \in \mathcal{Y}$ generated by a countable subset $\mathcal{C}_i \subset \mathcal{C}$. Consider the σ-algebra $\mathcal{B}$ generated by $\mathcal{C}_\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$. Then, by Proposition 1.7, $\mathcal{C}_\mathcal{B}$ is a countable subset of $\mathcal{C}$ because $\mathcal{C}_\mathcal{B}$ is a union of a countable collection of countable subsets $\mathcal{C}_i \subset \mathcal{C}$ $(i \in \mathbb{N})$. It follows that for each $i \in \mathbb{N}$, $\mathcal{B}_i \subset \mathcal{B}$, and moreover $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B} \subset \bigcup \mathcal{Y} = \mathcal{A}'$.

Let $X \in \mathcal{C}$. Then $\{X\}$ is a countable subset of $\mathcal{C}$, so that we may consider the σ-algebra $\mathcal{B}_X$ generated by $\{X\}$. It just so happens that $\mathcal{B}_X \in \mathcal{Y}$. Thus $X \in \mathcal{B}_X \subset \mathcal{A}'$ and so $\mathcal{A}'$ contains $\mathcal{C}$.

Now if $\mathcal{A}$ is the σ-algebra generated by $\mathcal{C}$, then $\mathcal{A} \subset \mathcal{A}'$. Furthermore, if $E \in \mathcal{A}$, then $E \in \mathcal{A}' = \bigcup \mathcal{Y}$ and therefore we must have $E \in \mathcal{A}_o$ for some $\mathcal{A}_o \in \mathcal{Y}$, i.e. $\mathcal{A}_o$ is generated by a countable subcollection $\mathcal{C}_o \subset \mathcal{C}$. $\square$