Sol'ns Ch6 and 11

6.2 #7 The easiest way to do these is to show they are a special case of something we've already proved. For example, if μ is counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, then $\mathcal{L}^p$ is just $\mathcal{L}^p(\mu)$, and so we can apply the Minkowski and Hölder results from Chapter 11. Since we officially didn't have these results by section 6.2, we have to find a way to apply Minkowski/Hölder for Lebesgue measure, which we did get in Ch.6. Given a sequence $\vec{\xi} = \langle \xi_n \rangle$ in ℓ^p, define a function $f_{\vec{\xi}} : \mathbb{R} \to \mathbb{R}$ by $f_{\vec{\xi}}(x) = \xi_n$ if $n \leq x < n+1$, 0 otherwise. One easily sees that the map $\vec{\xi} \mapsto f_{\vec{\xi}}$ takes ℓ^p into L^p (in fact, you did prove that, in proving Lemma 28 of Chapter 11), and that it preserves preserves the norm (i.e., is an *isometry*), so the Hölder and Minkowski inequalities for ℓ^p follow immediately from the corresponding results for Lebesgue measure on the line.

11.7 #44 First, note that if $x \in E_i$ then $f(x) = f_i(x)$, so $|f(x)|^p = |f_i(x)|^p = \sum_n |f_n(x)|^p$. It follows that $\|f\|^p = \int |f(x)|^p d\mu = \int \sum_n |f_n(x)|^p d\mu$, which by the MCT equals $\sum_n \int |f_n(x)|^p d\mu = \sum_n \|f_n(x)\|^p$. Thus, one is finite iff the other is.

Moreover, if they are finite, then $\|f - \sum_{i \leq n} f_i\|^p = \| \sum_{i>n} f_i\|^p \leq \sum_{i>n} \|f_i\|^p$ which $\to 0$ as $n \to 0$ since the series converges.

11.7 #48 The case when $p > 1$ is just what we did in class, and we know that even for $p = 1$ we have $\|F\| \leq \|g\|_q$. To show equality when $p = 1, q = \infty$, let's first assume that for any $M < \|g\|_\infty$ there is a set E_M such that $0 < \mu(E_M) < \infty$ and $|g| > M$ on E_M. (This is true for example for a sigma-finite measure space, or an atomless measure space.) Let $f_M = \frac{sign(g)}{\mu(E_M)} \chi_{E_M}$, note $\|f\|_1 = 1$, and

$$F(f_M) = \int f_M g d\mu = \frac{1}{\mu(E_M)} \int_{E_M} |g| d\mu > \frac{1}{\mu(E_M)} M \mu(E_M) = M$$

so $\|F\| \geq M$. Since $M < \|g\|_\infty$ is arbitrary, $\|F\| \geq \|g\|_\infty$.

Unfortunately, if $p = 1, q = \infty$, and the odd condition on the existence of sets E_M is *not* met, then in fact the result need not be true! For example, let $X = \{a, b\}$ be a space with just two points, $\mu(\{a\}) = 1, \mu(\{b\}) = \infty$, $g(b) = 5, g(a) = 0$, so $\|g\|_\infty = 5$. Note that if $f \in \mathcal{L}^p$ then $f(b) = 0$, so $F(f) = 0$. This means that $\|F\| = 0 \neq \|g\|_\infty$. (So, the text was a bit sloppy in stating this problem!)