

Non-text problems

1. Call a subset A of \mathbb{N} *cofinite* provided $\mathbb{N} \setminus A$ is finite. Show that the set of all finite and cofinite subsets of \mathbb{N} is an algebra, but is not a σ -algebra.
2. Call a subset A of \mathbb{R} *co-countable* provided $\mathbb{R} \setminus A$ is countable. Show that the set of all countable and co-countable subsets of \mathbb{R} is a σ -algebra.
3. (Hard!) Can there be a set Ω and a σ -algebra \mathcal{A} on Ω such that \mathcal{A} is countably infinite? (Hint: the answer is “no”. prove this.)
oo
4. Show that the version of Fatou’s Theorem in the text(11.3 Theorem 11) is equivalent to the one from class (“ILLLLL”) by giving a simple proof of each from the other.

5. This exercise outlines an alternate way of using the outer measure m^* we defined in class to get Lebesgue measure. So, suppose that $m^*(E)$ is defined to be the infimum of all sums of the form $\sum_{n=1}^{\infty} m(I_n)$ where I_n is a sequence of intervals with $E \subseteq \bigcup_n I_n$. We showed in class that (i) we may assume that the intervals I_n are open, and for any $\epsilon > 0$ we may assume that every $m(I_n) < \epsilon$; (ii) m^* is monotonic and σ -subadditive; (iii) If K_1, K_2 are disjoint compact sets then $m^*(K_1 \cup K_2) = m^*(K_1) + m^*(K_2)$; and (iv) $m^*(I) = m(I)$ if I is an interval.
- Show that for any bounded open set G and $\epsilon > 0$, there is a compact set K with $m^*(K) \geq m^*(G) - \epsilon$. (Hint: Every open set is a countable union of intervals, and every open interval can be approximated from within by closed intervals.)
 - Show that if K is compact, G is a bounded open set, and $K \subseteq G$, then $m^*(G \setminus K) = m^*(G) - m^*(K)$. (Hint: $G \setminus K$ is open.)
 - Call a set A *measurable* if for every $\epsilon > 0$ there is a compact set K and an open set G such that $m^*(G \setminus K) < \epsilon$. (Warning: this is an *alternative*, for the sake of this exercise, to the definition we gave in class/text.)
 - Show that any interval is measurable, and any *nullset* (i.e., set E with $m^*(E) = 0$) is measurable.
 - Show that the measurable sets form an algebra.
 - If A_n is a sequence of disjoint measurable sets and $A = \bigcup_n A_n$ then A is measurable, and $m^*(A) = \sum_n m^*(A_n)$.

Note that that does it – we’ve shown that the collection of measurable sets is a σ -algebra containing all intervals (hence all Borel sets), and that m^* is a measure when restricted to this σ -algebra.

6. Recall that $x \in \mathbb{R}$ is a *point of density* for $E \subseteq \mathbb{R}$ if

$$\lim_{h \rightarrow 0} \frac{m(E \cap [x-h, x+h])}{2h} = 1$$

and that we proved that for almost all $x \in E$, x is a point of density for E .

- (a) Call x is a *point of dispersion* for E if $\lim_{h \rightarrow 0} \frac{m(E \cap [x-h, x+h])}{2h} = 0$. Show that x is a point of density for E if and only if x is a point of dispersion for $\mathbb{R} \setminus E$.
- (b) Show that if $m(A \Delta B) = 0$ then every point of density for A is a point of density for B (and of course vice versa).
- (c) Define $\rho : \mathcal{B}_{\mathbb{R}} \rightarrow \mathcal{B}_{\mathbb{R}}$ by $\rho(A) := \{x \in \mathbb{R} : x \text{ is a point of density for } A\}$. Show that ρ is a Boolean homomorphism; that is, show that (i) $\rho(A \cap B) = \rho(A) \cap \rho(B)$, (ii) $\rho(\emptyset) = \emptyset$, and (iii) $\rho(A^c) = \rho(A)^c$. Also, $m(\rho(A) \Delta A) = 0$; in particular, ρ preserves measure.
- (d) Show that if $\{B_i\}_{i \in I}$ are all elements of the range of ρ , then $\bigcup_{i \in I} B_i$ is measurable. (The index set I need NOT be countable here!)

(The function ρ is sometimes called a *lifting*. Liftings are quite useful in measure theory. The nonempty elements of the range of a lifting satisfy the axioms of a topology, called the *lifting topology*.)

7. Let μ, ν be measures on (Ω, \mathcal{A}) . Prove that the following are equivalent:

- (i) $\mu \ll \nu$
- (ii) $\forall \epsilon > 0 \exists \delta > 0 \forall E \in \mathcal{A} \quad \nu(E) < \delta \Rightarrow \mu(E) < \epsilon$

(Hint on (i) implies (ii): Prove it by contradiction; for $\delta = 2^{-n}$ let E_n be the corresponding set witnessing failure, and put $E = \bigcap_N \bigcup_{n > N} E_n$)