Asst. 1 Sol'ns

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9:11 PM

① Let $\mathcal{Q} = \{E \subseteq X \mid E \text{ or } E^c \text{ finite}\}$.

$\emptyset$ finite $\therefore$ $\emptyset \in \mathcal{Q}$.

$E \in \mathcal{Q} \Leftrightarrow E$ or E^c is finite $\Leftrightarrow E^c$ or E is finite $\Leftrightarrow E^c \in \mathcal{Q}$, so
$\mathcal{Q}$ is closed under complements

Let $E, F \in \mathcal{Q}$. If E or F finite then $E \cap F$ is finite, so $E \cap F \in \mathcal{Q}$
If both E, F cofinite then $(E \cap F)^c = E^c \cup F^c$ finite, so $E \cap F \in \mathcal{Q}$
$\therefore$ $\mathcal{Q}$ is closed under $\bigcap_f$

$\therefore$ $\mathcal{Q}$ is an algebra.

If $\{x_n\}_{n \in \mathbb{N}}$ is infinite with no repetitions then $\forall n \; \{x_n\} \in \mathcal{Q}$ but

$\bigcup_n \{x_{2n}\}$ is neither finite nor cofinite, $\therefore$ $\mathcal{Q}$ is not a σ-algebra

③ Let $\mathcal{Q}$ be the algebra in question. Let $\mathcal{E}$ consist of sets of

the form $\bigcup_{i=1}^{n} [a_i, b_i)$ where $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$.

Let $\mathcal{D} = \{(-\infty, a) \cup E \cup [b, \infty) \mid a < b, E \in \mathcal{E}\}$.

Claim: $\mathcal{Q} = \mathcal{E} \cup \mathcal{D} \cup \{\emptyset, \mathbb{R}\}$.

$\supseteq$) $\mathcal{E} \subseteq \mathcal{Q}$ since each $[a_i, b_i) \in \mathcal{Q}$
$\mathcal{D} \subseteq \mathcal{Q}$ since if $E \in \mathcal{E}$ and $a < b$ then $(-\infty, a) \cup E \cup [b, \infty)$
$= E \cup [a, b)^c$
Of course, $\emptyset, \mathbb{R} \in \mathcal{Q}$

$\subseteq$) This follows if we can show $\underbrace{\mathcal{E} \cup \mathcal{D} \cup \{\emptyset, \mathbb{R}\}}_{=: \mathcal{C}}$ is an algebra,

which is tedious but not conceptually difficult:

- $\emptyset \in \mathcal{C}$ ✓
- Complements:
  $A = \emptyset$ or $\mathbb{R} \Rightarrow A^c = \mathbb{R}$ or $\emptyset \in \mathcal{C}$
  $A = \bigcup [a_i, b_i) \Leftrightarrow A^c = (-\infty, a_1) \cup [b_1, a_2) \cup \cdots \cup [b_n, \infty)$,
  so $A \in \mathcal{E} \cup \mathcal{D} \Leftrightarrow A^c \in \mathcal{D} \cup \mathcal{E}$
  $\therefore$ $\mathcal{C}$ is closed under complements
- Finite unions:
  Let $A, B \in \mathcal{C}$. If $A = \emptyset$ or $\mathbb{R}$ then $A \cup B = B$ or $\mathbb{R} \in \mathcal{Q}$.
  If $A, B \in \mathcal{E}$ then $A \cup B \in \mathcal{E}$
  If $A \in \mathcal{D}$ and $B \in \mathcal{D}$ or $\mathcal{E}$ then $A \cup B \in \mathcal{D}$

$\therefore$ so $\mathcal{C}$ closed under $\bigcup_f$ ✓