Solns Asst 2

Sunday, September 25, 2011
4:01 PM

1.2 #3

a) Let $A_0 = X$. Given A_n s.t. $\mathcal{M}\restriction_{A_n}$ infinite, let $E \in \mathcal{M}\restriction_{A_n} - \{\emptyset, A_n\}$, note either $\mathcal{M}\restriction_E$ or $\mathcal{M}\restriction_{(A_n - E)}$ or both is infinite, let $A_{n+1} = E$ or $(A_n - E)$ so that $\mathcal{M}\restriction_{A_{n+1}}$ is infinite. Then $A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \cdots \in \mathcal{M}$. Put $B_n = A_n - A_{n+1}$, then $\{B_n\}_{n\in\omega} \subseteq \mathcal{M}$ is a disjoint sequence of nonempty sets

b) If $I \subseteq \omega$ then $\bigcup_{i\in I} B_i \in \mathcal{M}$. Moreover, if $I, J \subseteq \omega$ and $i \in I \setminus J$ then $\emptyset \neq B_i \subseteq \bigcup_{i\in I} B_i - \bigcup_{i\in J} B_i$, so the function $I \mapsto \bigcup_I B_i$ is a 1-1 function from $\mathcal{P}(\omega)$ into $\mathcal{M}$, so $\operatorname{card}(\mathcal{M}) \geq 2^{\aleph_0}$.

1.2 #4

Let $\mathcal{A}$ be an algebra on X closed under countable increasing unions; we need to show that $\mathcal{A}$ is closed under arbitrary countable unions.

Given $\{A_n\}_{n\in\omega} \subseteq \mathcal{A}$ put $B_n = A_0 \cup \cdots \cup A_n$, $n \in \omega$. Since $\mathcal{A}$ is an algebra, $B_n \in \mathcal{A}$. Clearly $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$, so $\bigcup_n B_n \in \mathcal{A}$. Since $A_n \subseteq B_n = \bigcup_{i\leq n} A_i \subseteq \bigcup_{i\in\omega} A_i$, $\bigcup_n A_n \subseteq \bigcup_n B_n \subseteq \bigcup_n A_n$, so $\bigcup_n A_n \in \mathcal{A}$.

1.3 #7 By induction it suffices to show that if μ and ν are measures on $(X, \mathcal{M})$ and $a, b \in \mathbb{R}^+$ then $\eta = a\mu + b\nu$ is a measure on $(X, \mathcal{M})$:

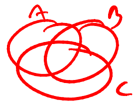
$$\eta(\emptyset) = a\mu(\emptyset) + b\nu(\emptyset) = a\cdot 0 + b\cdot 0 = 0 \checkmark$$

If $\{E_i\}_{i\in\omega} \subseteq \mathcal{A}$ is disjoint and $N \in \omega$ then

$$\sum_{i=0}^{N} \eta(E_i) = \sum_{i=0}^{N} (a\mu E_i + b\nu E_i) = a\sum_{i=0}^{N} \mu E_i + b\sum_{i=0}^{N} \nu E_i$$

Since μ and ν are measures, as $N \to \infty$ the right-hand side of this equation converges to $a\mu(\bigcup_{i\in\omega} E_i) + b\nu(\bigcup_{i\in\omega} E_i) = \eta(\bigcup_{i\in\omega} E_i)$, so the left-hand side also converges, so so the left-hand side also converges, so

$$\eta\left(\bigcup_{i\in\omega} E_i\right) = \sum_{i=0}^{\infty} \eta E_i \quad ✓$$

<u>1.3 #12</u> 

a) $E \cup F = (E \Delta F) \,\dot{\cup}\, (E \cap F)$, so $\mu(E\cup F) = \mu(E\Delta F) + \mu(E\cap F) = 0 + \mu(E\cap F) = \mu(E\cap F)$

Since $E\cap F \subseteq E, F \subseteq E\cup F$, $\mu(E\cap F) = \mu E = \mu F = \mu(E\cup F)$.

b) $\mu(E\Delta E) = \mu(\emptyset) = 0$; $\mu(A\Delta B) = 0 \Rightarrow \mu(B\Delta A) = 0$ since $A\Delta B = B\Delta A$;

If $A\sim B$ and $B\sim C$ then $A\Delta C \subseteq (A\Delta B)\cup(B\Delta C)$ (Why? Show!)

so $\mu(A\Delta C) \le \mu(B\Delta C) + \mu(A\Delta B) = 0+0 = 0$. ✓

c) The interesting part is the Δ inequality.

If $A, B, C \in \mathfrak{M}$ then (as above) $A\Delta C \subseteq (A\Delta B)\cup(B\Delta C)$ so

$\rho([A],[C]) = \mu(A\Delta C) \le \mu(A\Delta B) + \mu(B\Delta C) = \rho([A],[B]) + \rho([B],[C])$.

(Note: We should probably prove that ρ is well-defined. This follows since $\mu(A\Delta B)$ defines a <u>pseudometric</u> on $\mathfrak{M}$. We'll discuss this in class.)