Solutions

M631 extra Ch3 problems

(NOTE: Solutions below all assume all measures are positive.)

①

$\lambda \perp \mu \Rightarrow \exists A, B \quad A \cup B = X \;\&\; A \cap B = \emptyset \;\&\; \lambda A = \mu = \quad .$

Since $\mu B = 0$ and $\lambda \ll \mu$, $\lambda B = 0 \therefore \lambda(X) = \lambda(A \cup B) = \lambda(A) + \lambda(B) = 0$

②

a) $\mu \emptyset = \sum 2^{-n} \mu_n \emptyset = \sum 2^{-n} \cdot 0 = 0$

$\mu(\dot{\bigcup}_k A_k) = \sum_n 2^{-n} \mu_n (\dot{\bigcup}_k A_k) = \sum_n 2^{-n} \sum_k \mu_n A_k = \sum_k \sum_n 2^{-n} \mu_n A_k = \sum_k \mu A_k$

b) $\mu A = 0 \Rightarrow 2^{-n} \mu_n A \leq \sum_n 2^{-n} \mu_n A = \mu A = 0$, so $\mu_n A = 0$

c) If $E \in \mathcal{Q}$, $\int_E \sum_n 2^{-n} \frac{d\mu_n}{d\lambda} d\lambda \leq \sum_n 2^{-n} \int_E \frac{d\mu_n}{d\lambda} d\lambda$ (by MCT) $= \sum_n 2^{-n} \mu_n E = \mu E$

so $\sum_n 2^{-n} \frac{d\mu_n}{d\lambda} = \frac{d\mu}{d\lambda}$ a.e. by the uniqueness property of R-N derivatives.

③ $\lambda E = 0 \Rightarrow \mu E = \nu E = 0 \Rightarrow (\mu + \nu)(E) = 0 \therefore (\mu + \nu) \ll \lambda$

$\int_E \left(\frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda}\right) d\lambda = \int_E \frac{d\mu}{d\lambda} d\lambda + \int_E \frac{d\nu}{d\lambda} d\lambda = \mu E + \nu E = (\mu + \nu) E$, ✓

④ $\lambda E = 0 \Rightarrow \nu E = 0$ (since $\nu \ll \lambda$) $\Rightarrow \mu E \leq \nu E = 0$ so $\mu E = 0$

$\int_E \left(\frac{d\nu}{d\lambda} - \frac{d\mu}{d\lambda}\right) d\lambda = \int_E \frac{d\nu}{d\lambda} d\lambda - \int_E \frac{d\mu}{d\lambda} d\lambda = \nu E - \mu E \geq 0$ for any $E \in \mathcal{Q}$

$\therefore \frac{d\nu}{d\lambda} - \frac{d\mu}{d\lambda} \geq 0$, or $\frac{d\nu}{d\lambda} \geq \frac{d\mu}{d\lambda}$