Ch 6 Solutions

1. Fill in the following alternate proof of the Jensen inequality. (Please do *not* use the Jensen inequality in proving any of the steps!)

   (a) Suppose that ϕ is a convex function on an interval I, that $x_1, x_2, \ldots, x_n$ I, that $\lambda_1, \lambda_2, \ldots, \lambda_n \in (0,1)$, and that $\sum_i \lambda_i = 1$. Prove that $\phi(\sum_i \lambda_i x_i) \le \sum_i \lambda_i \phi(x_i)$

   (b) Use this to prove that for a simple function f on a measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) = 1$, $\phi(\int f d\mu) \le \int \phi \circ f d\mu$

   (c) Conclude that for an L_1 function f on such a space, $\phi(\int f d\mu) \le \int \phi \circ f d\mu$

a) Induct on n. This is trivial for $n=1$, and just the definition of convexity for $n=2$.

Suppose then that (a) holds for n. Let $x_1 \cdots x_{n+1} \in I$, $\lambda_1 \cdots \lambda_{n+1} \in (0,1)$, $\sum_{i=1}^{n+1} \lambda_i = 1$.

Put $\delta = \lambda_1 + \cdots + \lambda_n$, $\delta_i = \lambda_i / \delta$ for $i \le n$, note $\delta + \lambda_{n+1} = 1$, so

$$\varphi\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = \varphi\left(\delta \sum_{i=1}^{n} \delta_i x_i + \lambda_{n+1} x_{n+1}\right) \le \delta \varphi\left(\sum_{i=1}^{n} \delta_i x_i\right) + \lambda_{n+1} \varphi(x_{n+1}) \quad (*)$$

$$(**) \le \delta \sum_{i=1}^{n} \delta_i \varphi(x_i) + \lambda_{n+1} \varphi(x_{n+1}) = \sum_{i=1}^{n+1} \lambda_i \varphi(x_i).$$

The result follows for all n by induction.

Note that for $(*)$ we need that $\sum_{i=1}^{n} \delta_i x_i \in I$ (which follows since I is convex)

and $(**)$ follows from the induction hypothesis.

b) Let $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$. WOLG the sets E_i partition X (so some values of α_i will $=0$)

and for some $m \le n$, $i > m \Leftrightarrow \mu E_i = 0$.

Then $\varphi\left(\int f d\mu\right) = \varphi\left(\sum_{i=1}^{m} \alpha_i \mu E_i\right) \le \sum_{i=1}^{m} \varphi(\alpha_i) \mu E_i$ (by Jensen) $= \sum_{i=1}^{n} \varphi(\alpha_i) \mu E_i$

$= \int \sum_{i=1}^{n} \varphi(\alpha_i) \chi_{E_i} \, d\mu = \int \varphi \circ f \, d\mu$

c) Let $f \in L_1$. First, suppose f bounded, $|f| \le N$ on X. Since φ is continuous,

$\varphi([-N,N])$ is bounded, say $|\varphi| \le M$ on $[-N,N]$. Let f_n simple, $|f_n| \le |f| \le N$,

$f_n \to f$ pointwise. Note $\varphi(f_n) \to \varphi(f)$ pointwise, and $|\varphi(f)|, |\varphi(f_n)| \le M$.

Then $\varphi\left(\int f d\mu\right) = \varphi\left(\lim_n \int f_n d\mu\right)$ (by DCT) $= \lim_n \varphi\left(\int f_n d\mu\right)$ (since φ continuous)

$\le \lim_n \int \varphi(f_n) d\mu = \int \varphi(f) d\mu$ (by DCT again)

Now, if $f \in L_1$ is not bounded, but $\varphi(f) \in L_1$, then put $E_n = \{x \mid |f(x)| \le n, |\varphi(f)(x)| \le n\}$,

let $f_n = f \chi_{E_n}$, then $f_n \to f$, $|f_n| \le |f|$, $\varphi(f_n) \to \varphi(f)$, $|\varphi(f_n)| \le |\varphi(f)| + |\varphi(0)| \in L_1$,

so $\varphi\left(\int f d\mu\right) = \varphi\left(\lim \int f_n d\mu\right)$ (by DCT) $= \lim \varphi\left(\int f_n d\mu\right)$ (since φ continuous) $\le \lim \int \varphi \circ f_n d\mu$ (by above)

$= \int \varphi\circ f \, d\mu$ by DCT

Finally, suppose $\varphi\circ f \notin L_1$. Note that by convexity, $\varphi(0) \le \varphi(x) + M(x-0)$ for some M,

so $\varphi(x) \geq \varphi(0) - |M||x| \; \forall x$, or $\varphi(f(x)) \geq -|\varphi(0)| - |M||f(x)|$, so $0 \le (\varphi\circ f)_- \le |\varphi(0)| + |M||f(x)| \in L_1$.

It follows that $\int (\varphi\circ f)_+ \, d\mu = \infty$, so $\int (\varphi\circ f) \, d\mu = +\infty$, so again $\varphi\left(\int f d\mu\right) \le \int \varphi\circ f \, d\mu$.

2. Suppose that $f : [a,b] \to \mathbb{R}$ where $a, b \in \mathbb{R}$. Recall that

$$T_a^b(f) = \sup\{\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| \; : \; a = x_0 < x_1 < \cdots < x_n = b\}$$

$$P_a^b(f) = \sup\{\sum_{i=1}^{n-1} (f(x_{i+1}) - f(x_i))^+ \; : \; a = x_0 < x_1 < \cdots < x_n = b\}$$

$$N_a^b(f) = \sup\{\sum_{i=1}^{n-1} (f(x_i) - f(x_{i+1}))^+ \; : \; a = x_0 < x_1 < \cdots < x_n = b\}$$

Show that $T_a^b(f) = \infty$ if and only if *both* $P_a^b(f) = \infty$ and $N_a^b(f) = \infty$.
(You may use other facts from class about variation.)

$\Leftarrow$) Let $M > 0$. There is a partition $a = x_0 < \cdots < x_n = b$ s.t. $\sum_{i=1}^{n-1} (f(x_{i+1}) - f(x_i))^+ > M$.
Then $T_a^b(f) \geq \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| \geq \sum_{i=1}^{n-1} (f(x_{i+1}) - f(x_i))^+ > M$. Since M arbitrary,
$T_a^b(f) = \infty$. (Or just use the class fact that $P_a^b(f), N_a^b(f) \le T_a^b(f)$.)

$\Rightarrow$) Suppose $P_a^b(f) < \infty$. Let $a = x_0 < x_1 < \cdots < x_n = b$ be any partition of $[a,b]$,
$f(b) - f(a) = \sum_{i=1}^{n-1} \Delta f_i$ (where $\Delta f_i = f(x_{i+1}) - f(x_i)$) $= \sum_{\Delta f_i \geq 0} \Delta f_i - \sum_{\Delta f_i \le 0} -\Delta f_i$

$\therefore \sum_{\Delta f_i \le 0} -\Delta f_i = \sum_{\Delta f_i \geq 0} \Delta f_i + f(a) - f(b) \le P_a^b(f) + f(a) - f(b)$.

Since the partition was arbitrary, $N_a^b(f) \le P_a^b(f) + f(a) - f(b) < \infty$.

A similar argument shows that if $N_a^b(f) < \infty$, so is $P_a^b(f)$.

Finally, $T_a^b(f) = N_a^b(f) + P_a^b(f) < \infty$.