Solutions

Friday, October 28, 2011
10:14 PM

38 $\left|(f_n+g_n)(x)-(f+g)(x)\right| \le \left|f_n(x)-f(x)\right| + \left|g_n(x)-g(x)\right|$, so

$\mu\left(\{x \mid |(f_n+g_n)(x)-(f+g)(x)| \ge \varepsilon\}\right) \le \mu\left(\{x \mid |f_n(x)-f(x)| \ge \varepsilon/2\}\right) + \mu\left(\{x \mid |g_n(x)-g(x)| \ge \varepsilon/2\}\right)$

$\to 0$ as $n\to\infty$, $\therefore$ $f_n+g_n \to f+g$ in measure.

Now, suppose $\mu X < \infty$. Since $\{x \mid |f(x)| > m\} \downarrow \emptyset$ as $m \nearrow \infty$, $\mu([|f|>m]) \to 0$ as $m\to\infty$.

Similarly, $\mu([|g|>m]) \to 0$ as $m \to \infty$.

Since $|f_ng_n - fg| = |f_ng_n - f_ng + f_ng - fg| \le |f_n||g_n-g| + |g||f_n-f| \le (|f_n-f|+|f|)|g_n-g| + |g||f_n-f|$

for any $\varepsilon > 0$ and $M \in \mathbb{N}^+$ $\{x \mid |f_n(x)g_n(x) - f(x)g(x)| \ge \varepsilon\} \subseteq$

$\{x \mid |g(x)| > m\} \cup \{x \mid |f(x)| > m-1\} \cup \{x \mid |f(x)-f_n(x)| > \tfrac{1}{2}\} \cup \{x \mid |g_n(x)-g(x)| \ge \tfrac{\varepsilon}{3M}\} \cup \{x \mid |f_n(x)-f(x)| \ge \tfrac{\varepsilon}{3M}\}$

(since if $x \notin$ right-hand side, $|f_n(x)g_n(x) - f(x)g(x)| \le (\tfrac{1}{2}+m-1)(\tfrac{\varepsilon}{3M}) + m(\tfrac{\varepsilon}{3M}) < \varepsilon$)

Fix $\delta > 0$, let M large enough that $\mu([|f|>M-1])$ and $\mu([|g|>M])$ both $< \delta/5$,

let N large enough that $\forall n>N$ $\mu([|f-f_n| > \tfrac{1}{2}])$, $\mu([|f-f_n| \ge \tfrac{\varepsilon}{3M}])$, $\mu([|g-g_n| \ge \varepsilon/3M])$

are all $< \delta/5$, then for $n >$ this N, $\mu(\{x \mid |f_n(x)g_n(x) - f(x)g(x)| \ge \varepsilon\}) < 5\cdot\frac{\delta}{5} = \delta$,

proving $f_ng_n \to fg$ in measure.

If μX not $< \infty$, the result can fail. For example, let $(X, \mathcal{Q}, \mu)$ be

Lebesgue measure on $\mathbb{R}$, $f(x) = g(x) = x$, $f_n(x) = g_n(x) = x + 1/n$.

Then $\mu(\{x \mid |f_n(x)-f(x)| \ge \varepsilon\}) = \mu(\{x \mid 1/n \ge \varepsilon\}) = 0$ once $n > 1/\varepsilon$,

but $\mu(\{x \mid |f_n(x)g_n(x) - f(x)g(x)| \ge \varepsilon\}) = \mu(\{x \mid |\tfrac{2x}{n} + \tfrac{1}{n^2}| \ge \varepsilon\}) \ge \mu(\{x \mid |2x| \ge \varepsilon n\}) = \infty$ for all n

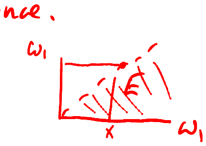
42 μ cty msre on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Show $f_n \to f$ in msre $\Leftrightarrow$ $f_n \to f$ uniformly

Let $\varepsilon > 0$ and $0 < \delta < 1$. The following statements are equivalent:

- $\exists N\ \forall n \ge N$ $\mu(\{x \mid |f_n(x)-f(x)| \ge \varepsilon\}) < \delta$
- $\exists N\ \forall n \ge N$ $\{x \mid |f_n(x)-f(x)| \ge \varepsilon\} = \emptyset$
- $\forall n \ge N$ $\forall x \in \mathbb{N}$ $|f_n(x)-f(x)| < \varepsilon$

This proves the equivalence.

47 WOLG $X = Y = \omega_1$.



For any $x < \omega_1$, $E_x = \{y \mid y < x\}$ countable since $x < \omega_1$ is a countable ordinal.

$E^y = \{x \mid x \geq y\} = \omega_1 - \{x \mid x < y\}$ is co-countable.

$\iint \chi_E \, d\mu \, d\nu = \int \mu(E^y) \, d\nu = \int 1 \, d\nu = 1$, $\iint \chi_E \, d\nu \, d\mu = \int \nu(E_x) \, d\mu = \int 0 \, d\mu = 0$

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$\iint |f| \, d(\mu\times\nu) = \sum_m \sum_n \chi_E(m,n) = \sum_m 2 = \infty$

where $E = \{(m,n) \mid m=n \text{ or } m=n+1\}$

$\iint f \, d\mu \, d\nu = \sum_{n=1}^{\infty} (1+-1) = \sum_{n=1}^{\infty} 0 = 0$, $\iint f \, d\nu \, d\mu = 1 + \sum_{m=2}^{\infty} (-1+1) = 1$