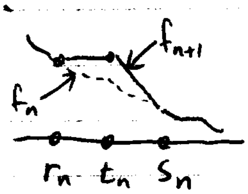
Example of a continuous function, $f(0) > f(1)$, but for no interval is f decreasing on the interval:

Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ enumerate subintervals of $[0,1]$ with rational endpoints, and let $f_0 : [0,1] \to \mathbb{R}$ be $f_0(x) = 1 - x$.

Given $f_n : [0,1] \to \mathbb{R}$, define f_{n+1} as follows:

If f_n is not strictly decreasing on (a_n, b_n), put $f_{n+1} = f_n$.
If f_n is strictly decreasing on (a_n, b_n), find r_n, s_n with $a_n < r_n < s_n < b_n$ and $f_n(r_n) - f_n(s_n) < 2^{-(n+1)}$. Let $t_n = (r_n + s_n)/2$.
Let

$$f_{n+1}(x) = \begin{cases} f_n(x) & \text{if } x \le r_n \text{ or } x \ge s_n \\ f_n(r_n) & \text{if } r_n \le x \le t_n \\ f_n(s_n) + \left(\dfrac{f_n(r_n) - f_n(s_n)}{t_n - s_n}\right)(x - s_n) & \text{if } t_n \le x \le s_n \end{cases}$$



Note: ① f_{n+1} is continuous; ② If f_n nondecreasing on some $[r,s]$, Then $f_i|_{[r,s]} = f_n|_{[r,s]}$ $\forall i > n$ ③ $\sup_{x \in [0,1]} |f_{n+1}(x) - f_n(x)| \le 2^{-(n+1)}$

④ For any x, $\{f_n(x)\}_n$ is a Cauchy sequence

From ④, $f(x) = \lim_{n \to \infty} f_n(x)$ exists for all x. From ③ one can show that f is continuous. If (for a contradiction) f is decreasing on some (a,b), Then f is decreasing on some (a_n, b_n).
If $f_n \downarrow$ on (a_n, b_n) then $f = f_{n+1} =$ constant on $[r_n, t_n]$, $\Rightarrow\Leftarrow$.
If $f_n \not\downarrow$ on (a_n, b_n) Then $\exists\, m+1$ smallest with f_{m+1} not decreasing on (a_n, b_n)
Then for some $r < s$, $a_n < r < s < b_n$, $f_{m+1}(r) \le f_{m+1}(s)$ but $f_m \downarrow$ on (a_n, b_n)
Then $[r,s] \subseteq [r_{m+1}, t_{m+1}]$, so $f_{m+1} =$ constant on $[r,s]$,
so $f =$ constant on $[r,s]$, $\Rightarrow\Leftarrow$.