5.1 #4

First, suppose $D^+f \geq \varepsilon$ on (a,b). Suppose (for a contradiction) that $\exists x<y$, $f(x)>f(y)$. WOLG $a=x$, $b=y$. Since f is continuous, $f(y)<f(a)$ for y in a neighborhood of b, so $\mathcal{C} := \{x \in (a,b) \mid f[[x,b]] \subseteq (-\infty, f(a))\} \neq \emptyset$; let $a' = \text{GLB of } \mathcal{C}$; $a'<b$. Since $D^+f \geq \varepsilon$, $f(a'+h) - f(a') \geq h(\frac{\varepsilon}{2})$ for arbitrarily small h, in particular for some h with $a'+h<b$. Then $a'+h \in \mathcal{C}$, so $f(a) > f(a'+h) \geq f(a') + h\varepsilon/2 \geq f(a')$, and by continuity of f $f(a) > f(x)$ for x in some neighborhood of a', contradicting choice of a'.

Now, if all we know is $D^+f \geq 0$, then $D^+g \geq \varepsilon$, where $g(x) = f(x) + \varepsilon x$. If $x<y$, $g(x) \leq g(y)$, so $f(x) + \varepsilon x \leq f(y) + \varepsilon y$. Since ε arbitrary, $f(x) \leq f(y)$ ✓

7b Let $\{a_n\}_n$ a nonnegative sequence with $\sum_n a_n < \infty$, let $\{r_n\}_n$ an enumeration of $\mathbb{Q} \cap [0,1]$.

let $f(x) = \sum \{a_n \mid r_n \le x\}$. Clearly f is nondecreasing on $[0,1]$.

If x rational, say $x = r_n$, and $y < x$, then $f(x) \geq f(y) + a_n$.

Then $\lim_{y \to x^-} f(y) \le f(x) - a_n \neq f(x)$, so f not continuous at x.

(Exercise: show that f is continuous at every $x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$!)

10 a) Let $a_k = \left(\frac{\pi}{2}(1+2k)\right)^{-1/2}$, $0 \le k$; for $x \in (0,1]$, $g(x) = \pm x^2$ iff $\exists k\ x = a_k$, and g is monotone on $[a_{k+1}, a_k]$ $\forall k$. It follows:

$$T_{-1}^{1} g \geq T_{a_N}^{1} g \geq \sum_{i=0}^{N-1} T_{a_{i+1}}^{a_i} g = \sum_{i=0}^{N-1} \left(\frac{2}{\pi}\right)\left(\frac{1}{1+2i} + \frac{1}{3+2i}\right) > \frac{1}{\pi} \sum_{i=0}^{N-1} \frac{1}{1+i} \to \infty \text{ as } N \to \infty,$$

so $T_{-1}^{1} g = \infty$.

b) Let $a_k = \left[\left(\frac{\pi}{2}\right)(1+2k)\right]^{-1}$, $0 \le k$; on $(0,1)$, $g(x) = \pm x^2$ iff $x = a_k$ for some k, and g is monotone on $[a_{k+1}, a_k]$ $\forall k$.

Let $0 = x_0 < x_1 < \dots < x_n = 1$ be any partition of $[0,1]$, N large enough that $a_N < x_0$, then

$$\sum_{i=0}^{n-1} |g(x_{i+1}) - g(x_i)| \le |g(x_0) - g(0)| + T_{x_0}^{1} g$$
$$= g(x_0)^2 + T_{a_N}^{1} g$$
$$\le 1 + T_{a_0}^{1} g + \sum_{i < N-1} \left(\frac{2}{\pi}\right)^2 \left(\frac{1}{(1+2i)^2} + \frac{1}{(3+2i)^2}\right) \le 1 + \sin 1 + \frac{8}{\pi^2} \sum_{i=0}^{\infty} \frac{1}{(1+2i)^2} < \infty.$$

Since the partition was arbitrary, $T_0^{1} g \le$ this stuff.

By symmetry, $T_{-1}^{0} g$ is also finite, so $T_{-1}^{1} g < \infty$.