Here is everything you really need to know about ordinals:

1. The definition: An ordinal number is a set \( \alpha \) such that (i) the elements of \( \alpha \) are well-ordered (by \( \epsilon \)), and (ii) (transitivity) if \( x \in y \in \alpha \) then \( x \in \alpha \)

2. It follows that every element of an ordinal \( \alpha \) is also a subset of \( \alpha \), and is in fact an ordinal itself.

3. Ordinals satisfy trichotomy, so any set of ordinals is linearly ordered.

4. There is a first ordinal (namely, \( \emptyset \)), but no last ordinal: in fact, if \( a \) is an ordinal, so is \( a \cup \{a\} \) (show this!), which we call the successor of \( a \), and denote by \( a^+ \). Note there is no ordinal strictly between \( a \) and \( a^+ \), so \( a^+ \) is the immediate successor of \( a \).

5. Any set of ordinals is well ordered; any nonempty collection of ordinals has a least element.

6. The natural numbers are (finite) ordinals:
   \[
   0 = \emptyset, 1 = \{\emptyset\} = \{0\}, 2 = \{0, 1\}, \ldots, n + 1 = \{0, 1, \ldots, n\}, \ldots
   \]

7. The set of natural numbers is also an ordinal, which we denote by \( \omega \) (so “\( n < \omega \)” is just a strange way of writing “\( n \in \mathbb{N} \)”). \( \omega \) is the least infinite ordinal, and is of course a countable ordinal.

8. There are many other countable ordinals. Examples include \( \omega^+ \), \( \omega + \omega \) (whatever that is), etc.

9. The first uncountable ordinal is denoted by \( \omega_1 \). To show that such an ordinal even exists requires a bit of work! In fact, \( \omega_1 \) is just the set of all countable ordinals.
10. Every well-ordered set is order-isomorphic to a unique ordinal.

11. By the Axiom of Choice, every set $X$ can be well-ordered, so has the same cardinality as some ordinal. The least such ordinal is the cardinality of $X$, or $\text{card}(X)$. The fact that every set has a cardinality is actually equivalent to AC. If $\kappa = \text{card}(X)$ then we can write $X = \{x_i\}_{i<\kappa}$, where all the elements $x_i$ are distinct.

12. (Extremely Useful) If $\{a_n\}_{n<\omega}$ is a set of countable ordinals, then for some $\beta < \omega_1$, $\forall n < \omega \ a_n < \beta$. 