Thm If $\mathcal{C} \subseteq \mathcal{P}(X)$ then there is a smallest σ-algebra $\mathcal{A}$ containing $\mathcal{C}$

Prf Define $\mathcal{C}_\alpha$, $\alpha < \omega_1$, by transfinite recursion:

$\mathcal{C}_0 := \mathcal{C} \cup \{\emptyset\}$

Given $\mathcal{C}_\beta$, $\beta < \alpha$, put

$$\mathcal{C}_\alpha := \Big(\bigcup_{\beta<\alpha} \mathcal{C}_\beta\Big) \cup \Big\{A^c \,\Big|\, A \in \bigcup_{\beta<\alpha} \mathcal{C}_\beta\Big\} \cup \Big\{\bigcup_{i\in\mathbb{N}} A_i \,\Big|\, \text{each } A_i \in \bigcup_{\beta<\alpha} \mathcal{C}_\beta\Big\}$$

Put $\mathcal{A} := \bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha$.

Note $\mathcal{C} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}_\alpha$ $\forall \alpha < \omega_1$, so $\mathcal{C} \subseteq \mathcal{A}$.

Similarly, $\emptyset \in \mathcal{A}$.

If $A \in \mathcal{A}$ then $\exists \alpha < \omega_1$ $A \in \mathcal{C}_\alpha$, so $A^c \in \mathcal{C}_{\alpha^+}$, where $\alpha^+ < \omega_1$ is the successor of α

If $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, then each $A_n \in \mathcal{C}_{\alpha_n}$ for some $\alpha_n < \omega_1$.

There is a $\beta < \omega_1$ such that $\alpha_n < \beta$ for all n.

Then each $A_n \in \mathcal{C}_\beta$, so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{C}_{\beta^+} \subseteq \mathcal{A}$

This proves that $\mathcal{A}$ is a σ-algebra containing $\mathcal{C}$.

To prove it is smallest, let $\mathcal{B}$ be any other σ-alg containing $\mathcal{C}$. Clearly $\mathcal{C}_0 \subseteq \mathcal{B}$. If $\mathcal{C}_\beta \subseteq \mathcal{B}$ for every $\beta < \alpha$, where α is some ordinal $< \omega_1$, then $\bigcup_{\beta<\alpha} \mathcal{C}_\beta \subseteq \mathcal{B}$, and since $\mathcal{B}$ is closed under complements and countable unions, $\mathcal{C}_\alpha \subseteq \mathcal{B}$ (from the def^n of $\mathcal{C}_\alpha$ above).

$\therefore$ $\mathcal{A} = \bigcup_\alpha \mathcal{C}_\alpha \subseteq \mathcal{B}$.