Final Exam – Math 632 – Spring 2007

INSTRUCTIONS: Do as many problems as possible, **but no more than 5**. Since the last problem was not assigned, I have made it worth more - that does not mean that it is the hardest problem, or requires the most work!
You may use notes and the class textbook.
You may not consult with anyone else, either in the class or outside. Discussing the problems on this exam in any way with anyone other than me for any reason before you hand it in is a form of cheating. (By "any reason" I mean any reason at all. For example, if you think there is a typo, *I* am the one to ask.)
This is a 2 hour exam, please try not to spend much more than that on these problems, and certainly no more than 3 hours altogether; I suggest you do it at one sitting.
The exam is due by 3:00 Friday. Bring your solutions to my office (slide under the door if I am not there) or email me your solutions.

(20) 1. Let d be the usual (Euclidean) metric $\mathbb{R}^n$. (a) Show that for any $\mathbf{z} \in \mathbb{R}^n$, $\{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{z}) \leq r\}$ is closed. (b) Is $\{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{z}) \leq r\}$ the closure of $\{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{z}) < r\}$? Justify your answer.

(20) 2. Let X be a metric space. Show that the sequence $\langle x_n \rangle_n$ has x as a cluster point if and only if there is a subsequence converging to x.

(20) 3. Let X be a totally bounded metric space, Y another metric space, and $f : X \to Y$ a continuous onto function. (a) Show that if f is uniformly continuous then Y is totally bounded. (b) Give an example where f is only continuous and Y is not totally bounded.

(20) 4. Let X be uncountable, and let $\tau = \{E \subseteq X : E^{\complement} \text{ is countable}\} \cup \{\emptyset\}$. (a) Show that τ is a topology. (b) Is this space Hausdorff? 2nd countable? 1st countable? (c) (5 points extra credit) Is the space Lindelöf?

(20) 5. Prove text page 160 Proposition 30

(20) 6. Prove that a connected subset of $\mathbb{R}$ having more than one point is an interval, and conversely that every interval in $\mathbb{R}$ is connected.

(25) 7. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $\mu(\Omega) = 1$, and f a measurable function.

a) Show that if $0 < r < s < \infty$ then $\|f\|_r \leq \|f\|_s$.

b) When does equality hold in (a)?

c) Show that (a) holds even if $s = \infty$