7.3.8 h is a rational function, so continuous on its range, including $[0,1)$.
$h'(x) = \frac{1}{(1-x)^2} > 0$ for $x \in (0,1)$, so h increasing on $[0,1)$, so h is 1-1.
If $y \geq 0$, then $x := \frac{y}{1+y} \in [0,1)$, and $h(x) = y$, so
h is onto $[0,\infty)$ with $h^{-1}(y) = y/(1+y)$, also rational $\therefore$ continuous
$\therefore$ h is a homeomorphism

If $\varphi: [0,1] \to [0,1)$ a homeomorphism, then since $[0,1]$ is compact
and φ continuous, $[0,1) = \varphi([0,1])$ is compact, but
$\{[0, 1-1/n)\}_{n \in \mathbb{Z}^+}$ is an open cover of $[0,1)$ with no finite
subcover, $\Rightarrow\Leftarrow$.

7.4.16 ① Suppose f continuous, and $x_n \to x$ in X. Let $U \subseteq Y$ open, $f(x) \in U$.
By continuity, $\exists V \subseteq X$, V open, $x \in V$. $\exists N\ \forall n \geq N\ x_n \in V$.
But then, $\forall n \geq N$, $f(x_n) \in U$, so $\lim\limits_{n\to\infty} f(x_n) = f(x)$.
② If f not continuous at x, $\exists \varepsilon > 0\ \forall \delta > 0\ \exists x_\delta \in B(x,\delta)$ s.t. $f(x_\delta) \notin B(f(x),\varepsilon)$.
Since $d_X(x, x_{1/n}) < \frac{1}{n} \to 0$, $\lim\limits_{n\to\infty} x_{1/n} = x$. Also, $d_Y(f(x_{1/n}), f(x)) \geq \varepsilon\ \forall n$,
so $\lim\limits_{n\to\infty} f(x_{1/n}) \neq f(x)$.
(or doesn't exist)

23 a) If $\varphi: X \to Y$ an isometry, $A \subseteq X$, Then
$\text{diam}(\varphi(A)) = \sup\{d_Y(\varphi(x), \varphi(y)) \mid x, y \in A\} = \sup\{d_X(x,y) \mid x, y \in A\}$,
so A bounded $\Leftrightarrow$ $\varphi(A)$ bounded.
To see it is not a uniform property, let $X = (\mathbb{R}, d_1)$, $Y = (\mathbb{R}, d_2)$
where d_1 = Euclidean metric, $d_2 = \min\{d_1, 1\}$.
Claim: id is uniformly continuous both ways, but doesn't preserve boundedness
If $\varepsilon > 0$ put $\delta < \min(\varepsilon, 1/2)$, Then $\forall x, y \in X$ $d_1(x,y) < \delta \Rightarrow d_2(x,y) = d_1(x,y) < \varepsilon$ ✓
Also, $\forall x, y \in Y$, $d_2(x,y) < \delta \Rightarrow d_1(x,y) = d_2(x,y) < \varepsilon$ so id, id^{-1} unif cont
However, X is bounded, Y is not

b) Suppose $\varphi: (X, d_X) \to (Y, d_Y)$ is a uniform homeomorphism, and
X is totally bounded. Fix $\varepsilon > 0$, $\exists \delta > 0$ $\forall x, y \in X$ $d_X(x,y) < \delta \Rightarrow d_Y(\varphi(x), \varphi(y)) < \varepsilon$.
There is a finite cover $B(x_1, \delta), \dots, B(x_n, \delta)$ of X by δ-balls.
If $y \in Y$ There are $x, x_i \in X$, $\varphi(x) = y$, $x \in B(x_i, \delta)$.
Then $d_Y(\varphi(x), \varphi(x_i)) < \varepsilon$, so $y \in \bigcup_i B(f(x_i), \varepsilon)$, so Y
is covered by $B(\varphi(x_1), \varepsilon), \dots, B(\varphi(x_n), \varepsilon)$.

c) $\tan\left(\frac{\pi x}{2}\right)$ is a homeomorphism from $(-1,1)$ to $\mathbb{R}$, $(-1,1)$ is $\underbrace{\text{tot. bounded}}_{\text{since } (-1,1) \subseteq [-1,1] \text{ compact, and compact} \Rightarrow \text{tot. bdd.}}$, $\underbrace{\mathbb{R} \text{ not}}_{\text{since tot. bdd} \Rightarrow \text{bdd}}$.

3.2.13 Let τ = $\{\emptyset\} \cup$ Closure of $\mathcal{B}$ under arbitrary unions. Since $X \in \mathcal{B} \subseteq \tau$, and
τ closed under $\bigcup$, τ a topology. Any topology containing $\mathcal{B}$ is closed under $\bigcup$,
so contains τ. Finally, $\mathcal{B}$ is a basis for τ: if $x \in U \in \tau$, $U = \bigcup_{i \in I} B_i$ for some
$\{B_i\}_{i \in I} \subseteq \mathcal{B}$, Then $\exists i \in I$ $x \in B_i \subseteq U$, done.

8.2.15 If $[a,b) \cap [c,d) \neq \emptyset$ then $[a,b)\cap[c,d) = [\max(a,c), \min(b,d)) \in \mathcal{B}$,

so $\forall A, B \in \mathcal{B}$, $x \in A \cap B \Rightarrow \exists C \in \mathcal{B}$, $x \in C \subseteq A \cap B$ (namely, $C = A \cap B$).

$\therefore \mathcal{B}$ is a basis (BTW, the topology $\mathcal{B}$ generates is the "Sorgenfrey Topology")

If $x \in \mathbb{R}$, $x \in [a,b)$, then $x < b$ $\therefore \exists n \in \mathbb{N}^+$, $[x, x+\frac{1}{n}) \subseteq [a,b)$

so $\{[x, x+\frac{1}{n})\}_n$ a countable basis at x $\therefore$ 1st countable

Suppose (for a contradiction) 2nd countable, then $\exists$ countable basis $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$.

For any x $\exists n = n(x)$ s.t. $x \in \mathcal{U}_n \subseteq [x, x+1)$; note x = least element of $\mathcal{U}_{n(x)}$,

so the function $x \mapsto n(x)$ is an injection of $\mathbb{R}$ into $\mathbb{N}$, $\Rightarrow\Leftarrow$.

If $\mathcal{U}$ open $\exists a < b$ with $[a,b) \subseteq \mathcal{U}$, and $\emptyset \neq \mathbb{Q} \cap (a,b) \subseteq [a,b) \cap \mathbb{Q} \subseteq \mathcal{U} \cap \mathbb{Q}$,

so $\mathbb{Q}$ is dense in $\mathbb{X}$, so $\mathbb{X}$ is separable.

Since metrizable + separable $\Rightarrow$ 2nd countable, 2nd countable + not separable $\Rightarrow$ not metrizable