Review of NSA and Ultrafilters

David A. Ross
Department of Mathematics
University of Hawai'i

August 14, 2017

Outline:

1. Comments on competing technologies
2. Absurdly quick review of nonstandard analysis
   (a) Def (not construction!) of a NS model
   (b) Transfer
   (c) Saturation
   (d) Internal sets
   (e) The hyperreals
   (f) More on saturation
3. Even quicker review of ultrafilters
   (a) Definition
   (b) Construction/Representation
   (c) Compactification
   (d) Multiplication
4. At least one proof (if time!)
5. Shameless plug for the Journal of Logic & Analysis

Competing technologies

Many of the asymptotic combinatorics results we're looking at can be done either using nonstandard analysis or directly with ultrafilters.

Logicians will say these are roughly the same thing, since nonstandard models are often constructed using ultraproducts.

This is a little misleading, since nonstandard models might also use other tools, like unions of chains.

The situation is analogous to geometric Banach space theory:
Krivine et al: ultraproducts of Banach spaces
Henson and Moore: nonstdandard hulls of Banach spaces

Some advantages of using NSA:

- Can simultaneously extend several objects at once, as well as easily understand the relationship between the extensions
- Can build complex properties easily into models
- Nonstandard proofs aren't always easy to translate into direct ultraproduct proofs
  (eg: Gordon Keller's 1972 theorem on uniform amenability)

Some advantages of *not* using NSA:

- Working directly with ultrafilters *feels* more concrete
- You can pretend you're not doing logic

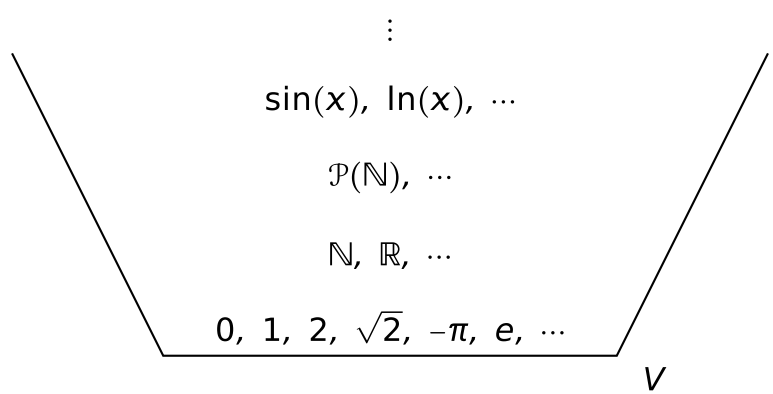
Components of a nonstandard model

(I) Start with a (standard) "mathematical universe" V:

- A large set containing every other mathematical object we might want to talk about, such as all natural numbers $0, 1, 2, \ldots$; real numbers $\sqrt{2}, -\pi, e, \ldots$; the *set* $\mathbb{N}$ of natural numbers *as an object*; the set $\mathbb{R}$ of real numbers; every function from $\mathbb{R}$ to $\mathbb{R}$, and the set of all such functions; etc.
- For example: use a **superstructure**:

$$V_0 := \mathbb{R}; \qquad V_{n+1} := V_n \cup \mathcal{P}(V_n); \qquad V := \bigcup_n V_n$$

- Pairs, relations, functions, etc. can be coded into V in the usual way.
- We call the elements of this mathematical universe **standard**.



(II) An infinite cardinal κ bigger than $\aleph_0$. It is often convenient to take $\kappa >$card(V).

(III) A first-order language $\mathcal{L}_V$ with constant, function, and relation symbols for every constant, function, and relation in V.

The nonstandard model consists of a new (bigger) superstructure ${}^*V$, and an injection

$$^*: V \to {}^*V$$

satisfying three properties:

1. Properness: If A is infinite then ${}^*A \supsetneq {}^\sigma A := \{{}^*a \ : \ a \in A\}$
2. Transfer
3. κ-saturation

Transfer: If S is an elementary statement about objects in V, then S is true in V if and only if its *star-transform* is true in ${}^*V$.

(Technically: every bounded-quantifier first-order $\mathcal{L}_V$-sentence ϕ holds in V if and only if it holds in ${}^*V$)

- *Bounded* means all quantifiers are bounded by elements of V:
- $(\exists x \in \mathbb{R})[x^2 = -1]$ is bounded, $(\exists x)[x^2 = -1]$ is not
- $(\forall A \subseteq \mathbb{N})(\exists x \in A)(\forall y \in A)[x \leq y]$ is not 1st order in $\mathcal{L}_V$,

  $(\forall A \in \mathcal{P}(\mathbb{N}))(\exists x \in A)(\forall y \in A)[x \leq y]$ <u>is</u> 1st order...
- ...provided one views quantifiers like $(\exists x \in A)$ as abbreviating $(\exists x \in \mathbb{N})[x \in A \& \ldots]$

So, since the well ordering property of the natural numbers is true in V,

$$(\forall A \in {}^*\mathcal{P}(\mathbb{N}))(\exists x \in A)(\forall y \in A)[x \,{}^*\!\leq y]$$

holds in ${}^*V$.

Transfer guarantees that the star embedding behaves the way one expects; for example,

- $*$ is one-to-one: if $a \neq b$ are standard elements, then $\neg(a = b)$ holds in V, so $\neg({}^*a = {}^*b)$ holds in ${}^*V$

- $*$ respects finite Boolean operations; for example, ${}^*(A \cup B) = {}^*A \cup {}^*B$ etc.

- ${}^*\mathcal{P}(A) \subseteq \mathcal{P}({}^*A)$ (but usually not equal!)

- $*$ respects ordered and unordered pairing: ${}^*\{a, b\} = \{{}^*a, {}^*b\}$, ${}^*\langle a, b\rangle = \langle {}^*a, {}^*b\rangle$, ${}^*A \times B = {}^*A \times {}^*B$, etc.

- If A is finite then so is ${}^*A$, and ${}^*A = \{{}^*a \ : \ a \in A\}$ $\quad (= {}^{\sigma}A)$

- If $f : A \to B$ is a (one-one/onto) function then so is ${}^*f : {}^*A \to {}^*B$

κ-Saturation Suppose that $\mathcal{S}$ is a collection of fewer than κ statements about an object X, and that for every finite subcollection of $\mathcal{S}$ there is an object in ${}^*V$ for which they hold; then there is an object in ${}^*V$ for which **all** the statements in $\mathcal{S}$ hold **at the same time**.

(Roughly means: Anything that can happen in ${}^*V$, does happen.)

Saturation is usually formulated in terms of a *internal sets*, a distinguished subcollection of sets of elements of ${}^*V$ (see next slide).

The nonstandard model is κ-saturated provided it satisfies:

Suppose $\{A_i\}_{i<\lambda}$ is a collection of internal sets such that

(a) $\lambda < \kappa$; and

(b) The collection $\{A_i\}_{i<\lambda}$ satisfies the *finite intersection property*, that is,

$$\text{for any } i_1 < i_2 < \cdots i_N < \lambda, \qquad (A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_N}) \neq \varnothing$$

Then $\bigcap_{i<\lambda} A_i \neq \varnothing$.

Internality

A subset $E \subseteq {}^*V$ is

Internal if $E \in {}^*A$ for some $A \in V$

External otherwise

The importance of internal sets can be seen in the transfer principle: if a quantifier $(\forall x \in A)$ appears in a sentence ϕ, then the corresponding quantifier in the star-transform of ϕ is $(\forall x \in {}^*A)$, that is, the quantifier ranges only over internal sets.

Internal sets play a role in nonstandard analysis comparable to measurable sets in measure theory and definable sets in logic.

Some external sets are important too!

Some internal sets:

1. If A in V then ${}^*A$ is internal.
2. If $C \in B$ and B internal then C is internal.
3. **(Internal definition principle)** If $\phi(x, y_1, \ldots, y_n)$ is a formula in $\mathcal{L}_V$ and $B, b_1, \ldots, b_n$ are internal, then

$$A = \{a \in B \mid \phi[a, b_1, \ldots, b_n] \text{ is true in } {}^*V\}$$

   is internal.

   It follows:

4. The internal sets are closed under finite boolean operations
5. If A is standard then ${}^*\mathcal{P}(A)$ is the (internal) set of internal subsets of ${}^*A$.
6. If A is internal then ${}^*\mathcal{P}(A)$ is the set of internal subsets of A (viewing $\mathcal{P}$ as a function defined on a set B where $A \in {}^*B$)

EG: If $a, b \in {}^*\mathbb{R}$ and $a {}^*\!< b$ then $(a, b) := \{s \in {}^*\mathbb{R} \mid a {}^*\!< s {}^*\!< b\}$ is internal.

(Note the starless parentheses)

Example: Consider the statements:

$$\text{“}x \text{ is a natural number”}, \text{“}x \geq 1\text{”}, \text{“}x \geq 2\text{”}, \text{“}x \geq 3\text{”}, \cdots$$

Any finite subset of these statements refers to a largest number N which satisfies this finite set of statements.

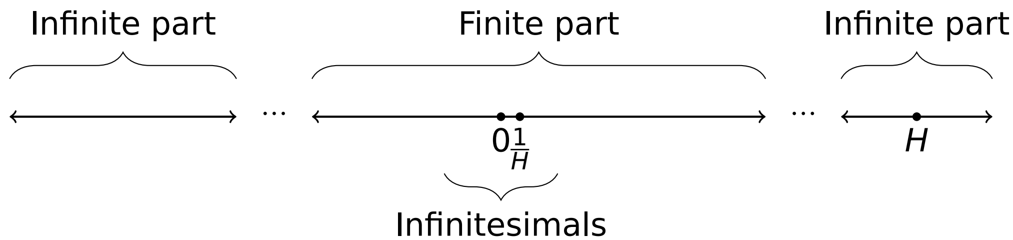
It follows that there is a an element H of ${}^*\mathbb{N}$ satisfying all the statements, that is, such that for every (standard) natural number n, $H > n$

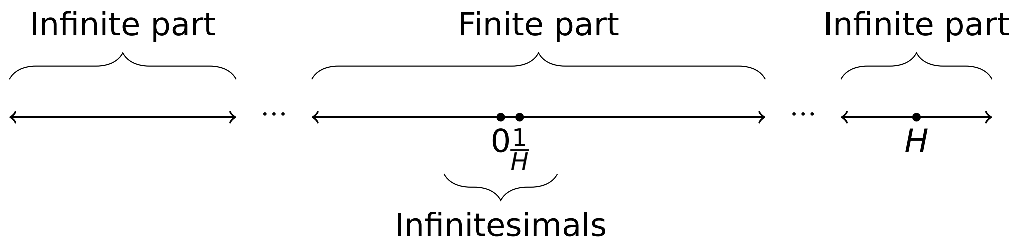
Such an H is an infinite *hyperfinite number*.

(Or note ${}^*[n, \infty)$ is internal for every $n \in \mathbb{N}$ and the sequence has the f.i.p., take $H \in \bigcap_{n=1}^{\infty} {}^*\mathbb{N} \cap {}^*[n, \infty)$ by $\aleph_1$-saturation)

Since $1/H$ is less than every standard real number, it is a *positive infinitesimal* in ${}^*\mathbb{R}$.

Since ${}^*\mathbb{R}$ (the set of *“hyperreal numbers*") is, like the usual set of real numbers, closed under the basic arithmetic operations, it is a non-Archimedean ordered field, and looks roughly like this:





In other words:

Infinitesimals: $monad(0) := \bigcap_{n=1}^{\infty} {}^*(-1/n, 1/n)$

Finite hyperreals: $ns(\mathbb{R}) := \bigcup_{n=1}^{\infty} {}^*(-n, n)$

Infinite hyperreals: $\bigcup_{n=1}^{\infty} \left({}^*(n, \infty) \cup {}^*(-\infty, -n)\right)$

$monad(0)$ and $ns(\mathbb{R})$ are ordered rings.

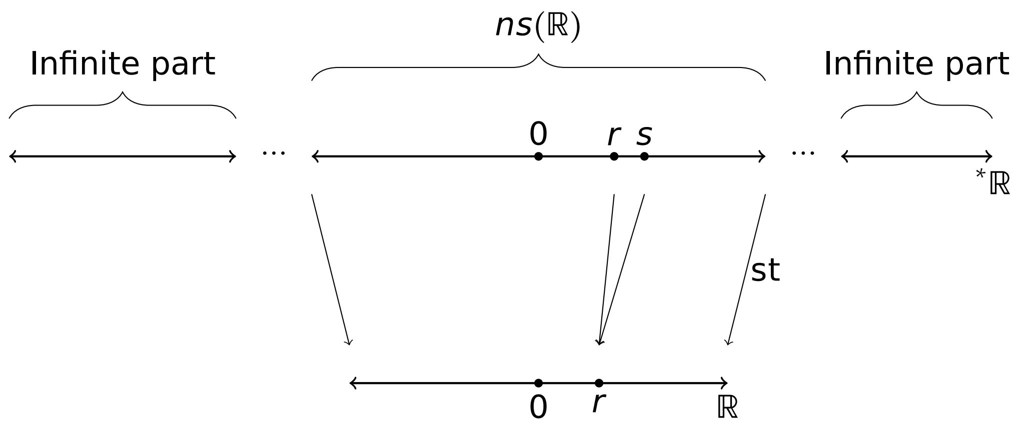
These are all external sets.

(Proof for $monad(0)$: If $monad(0)$ is internal then by transfer it has a LUB M in ${}^*\mathbb{R}$. If $M \in monad(0)$ then $M < 2M \in monad(0), \Rightarrow\Leftarrow$. If $M \notin monad(0)$ then $M > M/2 \notin monad(0), \Rightarrow\Leftarrow$.)

Write $r \approx s$ if $b - a$ is infinitesimal.

By completeness of $\mathbb{R}$ every finite hyperreal s is $\approx$ to some unique standard real r; call r the *standard part* of s, $r = \mathrm{st}(s)$ or $r = {}^{\circ}s$.

In other words, $\mathrm{st}()$ takes any finite hyperreal to the closest standard real number.



$\mathrm{st} : ns(\mathbb{R}) \to \mathbb{R}$ is a ring homomorphism.

(There are some technical details to be verified here, but they are all easy!)

For $r \in \mathbb{R}$ standard, write

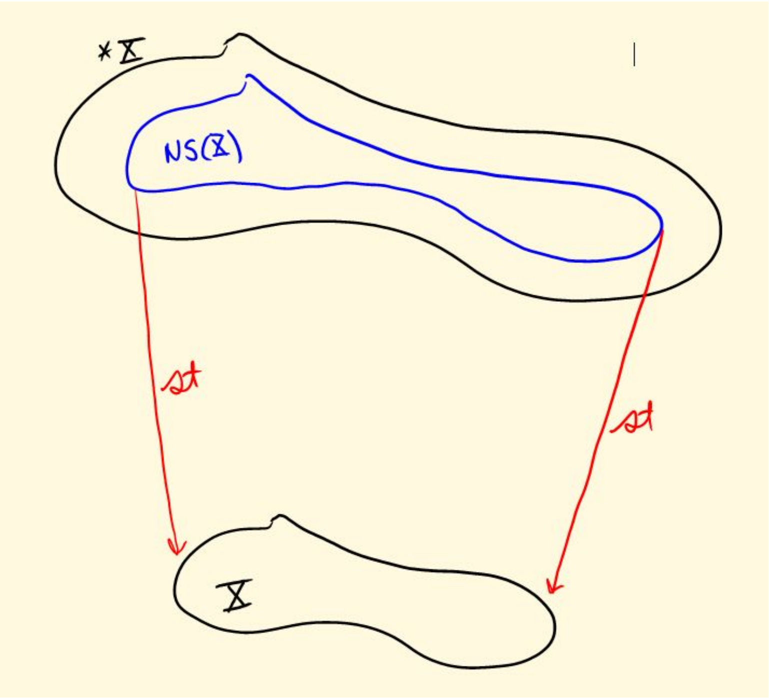
$$monad(r) = \{s \in {}^*R \;:\; r \approx s\} = \mathrm{st}^{-1}(r) = \bigcap\{{}^*u \;:\; u \text{ open}, r \in u\}$$

Digression: There is a similar situation for any reasonable nice topological space X.

For $x \in X$ write $monad(x) = \bigcap\{{}^*u : u \text{ open}, x \in u\}$ and $ns({}^*X) = \bigcup\{monad(x) : x \in X\}$

Topological properties can often be characterized in terms of these monads:
X is Hausdorff iff $x \neq y \Rightarrow monad(x) \cap monad(y) = \varnothing$.
X is compact iff ${}^*X = ns({}^*X)$
etc.

In the Hausdorff case we can define a standard part map, and the following picture makes sense:



The utility of κ-saturation was not immediately recognized. Early developments used weaker saturation properties that are interesting in their own right, eg for $A \subseteq \mathbb{R}$ internal,

Overflow: If A contains arbitrarily large finite numbers, then it contains an infinite number.

Overspill: If A contains every infinitesimal, then for some standard $r \in \mathbb{R}$, the interval $(-r, r) \subseteq A$

Underflow: If A contains arbitrarily small infinite numbers, then it contains a finite number.

Underspill: If A contains arbitrarily small nonifinitesimals, then it contains an infinitesimal.

(These only need the properness property and transfer, but are also easy consequences of κ-saturation.)

Another extremely useful property: If A is an infinite standard set, then there is a hyperfinite[1] $\hat{A}$ with ${}^{\sigma}A \subseteq \hat{A} \subseteq {}^{*}A$.

This one led Michael Richter to remark:

Nonstandard analysis is the art of making infinite sets finite by extending them.

The following uniform boundedness property, used by Kamae in his proof of Birkhoff's Ergodic Theorem, is just a reformulation of overflow:

If f is an internal function and $range(f) \subseteq \mathbb{N}$ (ie $f(x)$ is finite for all x) then f is bounded by some finite bound.

[1] (A hyperfinite means there is an internal bijection with $\{1, ..., H\}$ for some infinite integer H; that is, A has a *-finite internal *-cardinality.)

Other interesting saturation properties:

κ-enlarging property: If $\mathcal{F}$ is a standard set with the f.i.p. and $card(\mathcal{F}) < \kappa$ then $\bigcup\{^*F : F \in \mathcal{F}\} \neq \varnothing$

Isomorphism property/Special Model Axiom/Resplendancy: These do not follow even from κ-saturation with $\kappa > card(V)$, but can provides extra homogeneity in nonstandard objects. Introduced by Henson for Banach space theory.

Ultrafilters

Recall that an ultrafilter on a set S is a set $\mathcal{U} \subseteq \mathcal{P}(S)$ such that:

1. $\varnothing \notin \mathcal{U}$; $S \in \mathcal{U}$
2. If $A \subseteq B$ and $A \in \mathcal{U}$ then $B \in \mathcal{U}$
3. If $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$
4. (Ultra) For any $A \subseteq S$, either $A \in \mathcal{U}$ or $S \setminus A \in \mathcal{U}$

EG: If $\alpha \in S$ then $\mathcal{U}_\alpha := \{A \subseteq S : \alpha \in {}^*A\}$ is an ultrafilter

This is called the ultrafilter *generated by* α. Any filter on S generated by an element of S is called a principal ultrafilter.

Nonprincipal ultrafilters are generally “constructed" using some form of the Axiom of Choice.

EG: If $\alpha \in {}^*S \setminus S$ then $\mathcal{U}_s := \{A \subseteq S : \alpha \in A\}$ is a nonprincipal ultrafilter.

Note that if $\mathcal{U}$ is an ultrafilter on S and the model is sufficiently saturated then $\bigcap\{{}^*A : A \in \mathcal{U}\}$ is a nonempty set. Moreover, for any α in this set, $\mathcal{U} = \mathcal{U}_\alpha$.

Such sets partition ${}^*S$, and the partition classes can be identified with the space βS of all ultrafilters on S.

Stone-Čech Compactification

βS has a natural topology, where basic open sets have form
$U_A := \{B \subseteq S \,:\, A \subseteq B\}, \quad A \subseteq S.$

Obviously Hausdorff: If $A \in \mathcal{U} \setminus \mathcal{V}$ then $\mathcal{U} \in U_A, \mathcal{V} \in U_{S \setminus A}$

It is easy to see that if $\mathcal{U} \in {}^*\beta S$, ie is a *-ultrafilter on $^*S$, then

$$\{A \subseteq S \,:\, {}^*A \in \mathcal{U}\}$$

is an ultrafilter on A, in fact $\mathcal{U}$ is in its monad. Thus βS is compact.

When S has the discrete topology, the map $a \mapsto U_a$ is an obviously continuous injection of S into βS, so βS is a compactification of S.

In fact, βS is the *Stone-Čech* compactification of S:
If $f : S \to Y$, Y compact Hausdorff, then f lifts to a unique continuous map $\bar{f} : \beta S$ to Y, namely

$$\bar{f}(\mathcal{U}) := \mathrm{st}_Y({}^*f(a)), \quad \text{where } a \in {}^*S \text{ satisfies } \mathcal{U} = \mathcal{U}_a$$

Multiplication of ultrafilters

Suppose S is a discrete multiplicative semigroup, ie is equipped with an associative binary operation $\cdot$.

The “correct" multiplication on βS (in the sense of being useful for our purposes) is modeled on convolutions of measures.

If $\mathcal{U}, \mathcal{V}$ are ultrafilters on S, define $\mathcal{U} \odot \mathcal{V}$ by

$$A \in \mathcal{U} \odot \mathcal{V} \Leftrightarrow \{s \in S : s^{-1}A \in \mathcal{V}\} \in \mathcal{U}$$

where $s^{-1}A := \{t \in S : st \in A\}$.

(There might not be any actual element s^{-1}, of course, but if there is then $s^{-1}A$ is just the translation of A by s^{-1}.)

(Note to Mauro, Isaac, and Martino: I’m using s and t to be consistent with your monograph, but you might want to rethink this choice of variables given the ubiquity of st in nonstandard analysis...)

How is this a like a convolution? If we identify $\mathcal{U}$ with a $0-1$-valued finitely additive measure μ on S, likewise $\mathcal{U}$ with ν, and if μ_L (respectively, ν_L) is the Loeb extension (whatever that means) of μ (resp., ν) to $^*S$, then it can be shown that

$$A \in \mathcal{U} \odot \mathcal{V} \Leftrightarrow \iint \chi_{^*A}(st)d\nu_L(t)d\mu_L(s) = 1$$

$$A \notin \mathcal{U} \odot \mathcal{V} \Leftrightarrow \iint \chi_{^*A}(st)d\nu_L(t)d\mu_L(s) = 0$$