SOME NONSTANDARD MEASURE THEORY

DAVID A. ROSS

1. Introduction

This short introduction to nonstandard measure theory is a companion to the material I covered in my talk *Nonstandard measure theory and applications* at the 2012 UCLA Logic Summer School. While the talk arrived on schedule, these notes did not; I have completed them in hope that someone might find them useful. The imagined audience is a student who has learned some nonstandard analysis and who would like to see some nontrivial applications in measure theory.

While the original title was quite broad, I will focus on groups, mainly because I happen to like groups. This might provide a very distorted image of modern praxis in applied nonstandard analysis, as most of the most interesting standard applications of nonstandard measure theory (or *any* nonstandard analysis) over the last 30 years has been to probability (in the broader sense, including areas of stochastic processes, mathematical economics, and mathematical physics)[1]. While the word "probability" does appear in this outline, the subject does not. Any mathematics student genuinely interested in the field should learn some probability at the graduate level. This is true as well for the students not genuinely interested in the field.

This is basically an outline; I have left many proofs as exercises, have heavily self-plagiarized from earlier work and manuscripts-in-progress, and have omitted many of my favorite applications. I assume a basic familiarity with nonstandard analysis. The most comprehensive introduction is [2], but that is not widely available. At minimum I assume all the prefatory material from in Isaac Goldbring's notes on nonstandard analysis from the same UCLA Summer School. Another nice basic introduction is Martin Davis's book [3], though in many ways it is rather obsolete (compare the construction of Haar measure there to the one here).

2. Preliminaries

A nonstandard model starts with a *standard* model $\mathbb{V}$ containing all the mathematical objects one might want to discuss, an extension ${}^*\mathbb{V}$, and an injection ${}^* : \mathbb{V} \to {}^*\mathbb{V}$. As usual, we will assume the *transfer property* (essentially, ${}^*\mathbb{V}$ is an elementary extension of $\mathbb{V}$) and a high-but-unspecified amount of saturation. The model ${}^*\mathbb{V}$ has a distinguished family of objects, the *internal* sets, which includes the range of *.

Date: Original version 3/2013; this version 9/2017.

2000 *Mathematics Subject Classification.* Primary 28E05; Secondary 26E35, 28A20.

Key words and phrases. Loeb measure, nonstandard analysis, group, amenable.

[1] This is now changing! The last few years — since these notes were first written — has seen substantial growth of interest in applications of Loeb measures to combinatorics, additive number theory, pure model theory, and even groups.

(Because * is a function, we usually write $^*A$ instead of A^*; this has been the usual convention since the time of Robinson. Note however that the Goldbring notes adopt the opposite convention.)

In the 1970s the notion of *nonstandard hull* was introduced by Luxemburg. If X is an internal metric space then $Fin(X)/\approx$ (where $Fin(X)$ is the set of elements of X a standardly finite distance from some fixed element of X) is a standard, complete metric space. (In fact, if Y is a *standard* metric space, then $Fin(^*Y)/\approx$ is its completion.) This space is called a nonstandard hull of X. The construction can be applied to spaces other than metric spaces, for example topological spaces. When it is applied to measure spaces we call it the *Loeb construction*.

It can't be overemphasized how much of a change this was to the practice of nonstandard analysis. Prior to the introduction of nonstandard hulls, nonstandard analysis was always about objects in $\mathbb{V}$ or $^*\mathbb{V}$. Hulls are new objects that live in neither model. We note that in some 'flavors' of nonstandard analysis this is not possible, and if you are interested in hulls (or more generally in measure theory or functional analysis) you should avoid developments using alternate set theories (such as *Internal Set Theory*, IST).

3. Measures

Recall that a *measure space* is a triple $(X, \mathcal{A}, \mu)$ such that $\mathcal{A}$ is a $\sigma-$algebra on X and μ is a countably-additive measure on $(X, \mathcal{A})$. If we only assume that $\mathcal{A}$ is closed under finite Boolean operations (not all countable ones) and μ is only finitely-additive, then $(X, \mathcal{A}, \mu)$ is a *finitely-additive measure space*. (In characteristic oddball mathematical fashion, adding the qualifier "finitely-additive" to the description *expands* the class.)

Consider then the general question:

How does one use nonstandard analysis to build a σ-additive measure?

The fundamental difficulty involved in this question is reflected in the following fact.

Lemma 3.1. *Let $(X, \mathcal{B})$ be a standard measurable space and μ an internal $^*-$fa $^*-$measure on $(^*X, {}^*\mathcal{B})$. Then:*

(1) $^\circ\mu$ is a fa measure on $(^*X, {}^*\mathcal{B})$; and
(2) $E \mapsto {}^\circ\mu(^*E)$ is a fa measure on $(X, \mathcal{B})$.

However, even if μ is a measure and $\mathcal{B}$ is a $\sigma-$algebra, $^*\mathcal{B}$ need not be a $\sigma-$algebra (it generally won't be!) and $^\circ\mu$ need not be $\sigma-$additive.

This restriction meant that "prehistoric" nonstandard measure theory was mainly about representing and extending measures. For example:

Theorem 3.1. *(Bernstein, Wattenberg 1969; Henson 1972) If $(X, \mathcal{B}, \mu)$ is an atomless fa measure space, then there is a hyperfinite $\Omega \subseteq {}^*X$ such that for every $A \in \mathcal{B}$, $\mu(A) = {}^\circ(\|^*A \cap \Omega\| / \|\Omega\|)$*

Exercise 3.1. *Prove it. (Hint: use card$(\mathcal{B})^+$ saturation.)*

This was used by Bernstein and Wattenberg to give a simple proof of Banach's result that there is a translation/rotation-invariant measure on $\mathbb{R}$. (It is worth trying this without looking up their paper!)

A slightly simpler example is the construction of an invariant probability measure on $\mathbb{Z}$:

Example 3.1. *Let $H \in {}^*\mathbb{N}$ infinite, $I = [-H, H] \subseteq {}^*\mathbb{R}$, and for $A \subseteq \mathbb{Z}$ put $\mu(A) = {}^\circ(\frac{\|{}^*A \cap I\|}{2H+1})$. Then μ is a fa probability measure on $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$, and for any $A \subset \mathbb{Z}$ and $a \in \mathbb{Z}$ we have*

$$|\mu(a + A) - \mu(A)| \approx |\frac{\|{}^*(a + A) \cap I\|}{2H + 1} - \frac{\|{}^*A \cap I\|}{2H + 1}| \leq \frac{2|a|}{2H + 1} \approx 0$$

Thus μ is a $\mathbb{Z}$-invariant fa probability measure on the additive group $(Z, +)$

(Here *probability* means that the total measure is 1.)

Keep this example in mind as we proceed into the next section (on groups). However, before going there I want to give two more examples of prehistoric applications of nonstandard analysis, because they are fun. They both involve ultraflters. I will present them as exercises, and leave the details to you.

Exercise 3.2. *If $H \in {}^*\mathbb{N}$ then $U_H = \{A \subseteq \mathbb{N} \ : \ H \in {}^*A\}$ is an ultrafilter. U_H is nonprincipal if and only if H is infinite. Every ultrafilter on $\mathbb{N}$ is U_H for some H. (How is this related to the Bernstein-Wattenberg-Henson result?)*

Exercise 3.3. *Fix an infinite $H \in {}^*\mathbb{N}$, and let $E = \{x \in (0, 1) \ : \ \text{if } x = 0.d_1d_2d_3\cdots$ is a dyadic expansion of x, then $d_H = 1\}$ Then E is not Lebesgue-measurable. (Hint: Show that E cuts every open set in half, so by outer-regularity must cut itself in half.)*

3.1. **Loeb measures.** In general, we are interested in $\sigma-$additive measures. The breakthrough in nonstandard measure theory was a measure construction introduced by Peter Loeb in the early 1970s, and now called the *Loeb measure construction*. The idea behind this powerful tool is remarkably simple.

Let $(X, \mathcal{A}, \mu)$ be an internal finitely additive finite *-measure. This means that:

- X is an internal set
- $\mathcal{A}$ is an internal $^*-$algebra on X
- $\mu : \mathcal{A} \to {}^*[0, \infty)$ is an internal function satisfying (i) $\mu(\emptyset) = 0$, (ii) $\mu(X)$ is finite, and and (iii) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$ are disjoint.

Note that $\mathcal{A}$ is (externally) an algebra on X, and $st \circ \mu = {}^\circ\mu$ is an "actual" finitely-additive measure on $(X, \mathcal{A})$.

If $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ is a sequence of elements of $\mathcal{A}$ indexed by the standard natural numbers, and the intersection $\bigcap_n A_n$ is empty, then by $\aleph_1-$saturation there is a finite N such that $\bigcap_{n \leq N} A_n = \emptyset$. ($\therefore {}^\circ\mu$ is $\sigma-$additive on $\mathcal{A}$.)

The Carathéodory extension criterion is therefore satisfied trivially, and $(X, \mathcal{A}, {}^\circ\mu)$ extends to a countably-additive measure space $(X, \mathcal{A}_L, \mu_L)$, where $\mathcal{A}_L$ is the smallest (external) sigma-algebra containing $\mathcal{A}$. We call this the *Loeb space* built from $(X, \mathcal{A}, \mu)$.

The Loeb measure is a standard measure, so the integration theory one studies in a graduate analysis course applies. At the same time, this measure is strongly connected to the original internal *-measure. For example, if $E \in \mathcal{A}_L$, and $\epsilon > 0$ is standard, then $\exists A_i, A_o \in \mathcal{A}$ such that $A_i \subseteq E \subseteq A_o$ and $\mu(A_o) - \mu(A_i) < \epsilon$. Similarly, if $f : X \to \mathbb{R}$ is internally measurable (as any internal f would be in the common case where $\mathcal{A} = {}^*\mathcal{P}(X)$) and standardly bounded, then ${}^\circ f$ is a μ_L-measurable function.

For a complete introduction to/survey of nonstandard measure theory, see [5].

4. Groups

I prefer the notation of multiplicative groups to additive ones, so from now on G will always denote a multiplicative group with identity e.

A set $S \subseteq G$ *generates* a group G if G is the smallest subgroup of G which contains S; equivalently, the intersection of all subgroups of G that contain S. If a set S is *symmetric*, that is, $e \in S = S^{-1} =: \{g^{-1} : g \in S\}$, then it is easy to see that S generates G provided $G = \bigcup_{n \in \mathbb{N}} S^n$. G is *finitely generated* provided there is a finite S which generates G.

The group G is *amenable* if there is a nontrivial, left-invariant finitely-additive measure μ on $(G, \mathcal{P}(G))$ with $\mu(G) = 1$. (Call such a μ a *mean*.)

We have seen that $\mathbb{Z}$ is amenable under addition. One can show that the class of amenable groups includes all Abelian groups, all finite groups, homomorphic images and subgroups of amenable groups, etc. (That finite groups are amenable is a trivial exercise that you should do. In your head. Right now. The proof that Abelian groups are amenable follows easily from an argument like that for $\mathbb{Z}$ together with a thing called the Structure Theorem for Abelian Groups, and is a challenging and fun exercise, especially if you don't know this structure theorem.)

Not all groups are amenable:

Example 4.1. *Let F_2 be the free group on two generators, i.e., all formal words on the letters $\{a, b, a^{-1}, b^{-1}\}$ with the only simplification rules being the obvious ones. Then F_2 is not amenable.*

Proof. Let A consist of all elements of F_2 starting with a^n for some $0 \neq n \in \mathbb{Z}$, and let $B = F_2 \setminus A$. Suppose F_2 was amenable with G-invariant measure μ. Since $aB \subseteq A$ and $bA \subseteq B$, $\mu(A) = \mu(B) = 1/2$. On the other hand, the sets B, aB, a^2B are all disjoint with the same measure, so $1 \geq \sum_{k=0}^{2} \mu(a^k B) = 3/2$, a contradiction. ⊣

The word "amenable" and the definition above are due to Mahlon Day in the 1950s, but the idea originated with John von Neumann in 1929, who showed that F_2 is not amenable. von Neumann's characterization is the one given in the following theorem.

There are many useful equivalents for amenability. For some of them the Loeb measure is useful, so I suggest you go back and review the definition/construction of the Loeb measure.

Theorem 4.1. *Let (G, e) be a multiplicative group. Denote by $L^\infty(G)$ the bounded real functions on G. The following are equivalent: (a) G is amenable; (b) there is a positive linear functional $T : L^\infty(G) \to \mathbb{R}$ which is G-invariant in the sense that for any $g \in G$ and $f \in L^\infty(G), T(f) = T(f \circ \phi_g)$, where ϕ_g is the function $a \mapsto ga$ from G to G.*

Proof. Suppose first that μ is a mean on G. Apply the Loeb construction to the internal measure $({}^*G, {}^*\mathcal{P}(G), {}^*\mu)$ to obtain a standard probability measure $({}^*G, {}^*\mathcal{P}(G)_L, \mu_L)$. If $f \in L^\infty(G)$ then ${}^{\circ *}f$ is in $L_1(\mu_L)$, so we can define a functional by $T(f) = \int_{{}^*G} {}^{\circ *}f d\mu_L$. Since $f \mapsto {}^{\circ *}f$ is linear, and integration is a positive linear functional, T is a positive linear functional. If $g \in G$ then $T(f \circ \phi_g) \approx \int_{{}^*G} ({}^{\circ *}f \circ \phi_g) d\mu_L = \int_{{}^*(g^{-1}G)} {}^{\circ *}f d\mu_L \approx T(f)$ since $g^{-1}G = G$.

In the other direction, if T is an invariant positive linear functional on G then $\mu(A) = T(\chi_A)$ is a mean on G. ⊣

You are invited to try to prove $(a) \Rightarrow (b)$ without using the Loeb measure (or something similar, like a limit through an ultrafilter). Obviously it is possible (von Neumann did it!), but it is far from the near-triviality of the argument here.

As with any 1st-order structure, if G is a group, then ${}^*G$ is not only a *group, but also a group. Does amenability of one imply amenability of the other?

Theorem 4.2. *If ${}^*G$ is amenable then so is G. In fact, if there is an amenable subgroup G' of ${}^*G$ with $G \subseteq G'$ then G is amenable*

Of course, the hypothesis here is that ${}^*G$ is *externally* amenable. That is, there is a measure $\mu : \mathcal{P}({}^*G) \to \mathbb{R}$ such that

$$(\forall g \in {}^*G)[\mu(E) = \mu(aE)]$$

Proof. Let μ be an mean on G'. Define ν on $(G, \mathcal{P}(G))$ by $\nu(A) = \mu(G' \cap {}^*A)$, it is easy to verify that ν is a mean. ⊣

Is there a converse? *No!* We will see an example later.

Another quite useful equivalence of amenability is the following *Følner Condition*:

Theorem 4.3. *(Følner): G is amenable if and only if:*

$$\forall A \subseteq G \text{ finite } \forall \epsilon > 0 \; \exists E \subseteq G \text{ finite } \forall a \in A \; \frac{\|E \triangle aE\|}{\|E\|} < \epsilon$$

E is called a *Følner set*. This characterization of amenability is motivated by the example above of $\mathbb{Z}$, where $[-H, H] \cap {}^*\mathbb{Z}$ provided an internal Følner set that worked for all standard ϵ and A. It will occasionally be useful to apply this condition in its equivalent form,

$$\forall A \subseteq G \text{ finite } \forall r < 1 \; \exists E \subseteq G \text{ finite } \forall a \in A \; \frac{\|E \cap aE\|}{\|E\|} > r$$

Proof. We'll just prove sufficiency of the Følner condition; the proof is due to Henson [4]. Assuming the condition, by saturation there is a hyperfinite $E \subseteq {}^*G$ such that for every standard ϵ and $a \in G$, $\frac{\|E \triangle aE\|}{\|E\|} < \epsilon$. Since ϵ is arbitrary, ${}^\circ(\frac{\|E \triangle aE\|}{\|E\|}) = 0$. Put $\mu(A) = {}^\circ(\frac{\|E \cap {}^*A\|}{\|E\|})$ for $A \subseteq G$; it is easy to verify that μ is an invariant measure. ⊣

Theorem 4.4. *Let G be a group. TFAE:*

(1) G is amenable
(2) Every subgroup of G is amenable
(3) Every finitely-generated subgroup of G is amenable

Proof. $(2 \Rightarrow 3)$ is trivial. $(3 \Rightarrow 1)$ Let S be a hyperfinite subset of ${}^*G$ which contains G, let S' and let $\hat{G}$ be the internal ${}^*-$group generated by S. By transfer $\hat{G}$ is ${}^*-$amenable; let μ be a ${}^*-$mean. For $A \subseteq G$ let $\nu(A) = {}^\circ\mu({}^*A)$. It is easy to verify that μ is a mean on G.

$(1 \Rightarrow 2)$ This is standard but interesting, so I include it for completeness. If μ is a mean G, and H is a subgroup of G, note G can be partitioned into a disjoint union of cosets $Hg_i, i \in I$ for some index set I. For $A \subseteq H$ put $\nu(A) = \mu(\bigcup_{i \in I} Ag_i)$. Since this union remains disjoint it is easy to show that ν is a fa probability measure on G. For invariance, let $a \in H$ and $A \subseteq H$, then $\nu(aA) = \mu(\bigcup_{i \in I}(aA)g_i) =$

$\mu(a\bigcup_{i\in I} Ag_i) = \mu(\bigcup_{i\in I} Ag_i) = \nu(A)$, where the second equality uses the fact that $aAg_i \subseteq Hg_i$ for all i.

$\dashv$

We now return to the question above, as to whether amenability of G implies amenability of ${}^*G$.

Example 4.2. *Let $G = \{\pi \in Permutations(\mathbb{N}) : \exists N \in \mathbb{N}\ \forall x > N\ \pi(x) = x\}$. Then G is amenable, but ${}^*G$ is not.*

Before we prove this, let's consider how this fits with many other results. From nonstandard Calculus we know that if a function on $\mathbb{R}$ satisfies the $\epsilon - \delta$ definition of continuity, its star on $\mathbb{R}$ might not (with the standard ϵ and δ). In fact, if ${}^*f$ is S-continuous (satisfies the $\epsilon - \delta$ definition of continuity with standard ϵ and δ) then f is *uniformly* continuous. This uniformity extends to *sets* of functions. It would therefore be natural to call a group *uniformly amenable* if ${}^*G$ is (standardly) amenable.

Proof. **Claim 1:** G is amenable. One way to see this is to note that every finitely-generated subgroup of G is finite, so trivially amenable, and by Theorem 4.4 this implies that G is amenable. Or, use Følner: Let $A = \{a_1, \dots, a_n\} \subset G, r < 1$. For some sufficiently large N and all $x > N$ and $i \le n,\ a_i(x) = x$. let $E = \{0, \dots, M\}$, where $M > (N + r + 1)/(1 - r)$. Now, if $a \in A$ then $E \cap aE \supseteq \{N + 1, \dots, M\}$, so $\frac{|E\cap aE|}{|E|} \ge \frac{(M-N-1)}{M+1} > r$ by the choice of M.

Claim 2: ${}^*G$ is not amenable. It suffices to find $F_2 \subseteq {}^*G$, where F_2 is freely generated by $\{a, b\}$. Let $M \in {}^*\mathbb{N} \setminus \mathbb{N}$, let $\hat{F}$ be the (internal) set of all words of length at most M from $\{a, b, a^{-1}, b^{-1}\}$. Write $\hat{F} = \{f_0, \dots, f_{H-1}\}$ (where H is the internal cardinality of $\hat{F}$, and $f_0 = e$), and identify this set with $\{0, \dots, H - 1\}$.

Let $\hat{F}_a = \{g \in \hat{F} | ag \in \hat{F}\}$, and $\hat{F}_b = \{g \in \hat{F} | bg \in \hat{F}\}$. There is an internal bijection $\hat{a} : \hat{F} \to \hat{F}$ such that $\hat{a}(g) = ag$ for every $g \in \hat{F}_a$. Same for $\hat{b}$. Note $F_2 \subseteq \hat{F}_a \cap \hat{F}_b$. By the identification above, $\hat{a}, \hat{b} \in {}^*G$. Claim: if $w(x, y)$ is a word and $f_i \in F_2$, then $w(a, b)f_i = f_{w(\hat{a},\hat{b})(i)}$. The proof is an easy induction on the length of w. It follows that if $w(\hat{a}, \hat{b}) = id$ then $w(a, b) = e$, and this proves that $\hat{a}$ and $\hat{b}$ generate a free group. $\dashv$

5. Keller's Theorem

We now prove a remarkable theorem of Keller that says roughly that if every group in a "nice" class is amenable, then the class is uniformly amenable. Of course, we need to define "nice" and "uniformly amenable".

A word $w(x_1, x_2, \dots, x_n)$ is an *identity relation* (or *law*) for G provided $\forall a_1, \dots, a_n \in G, w(a_1, \dots, a_n) = e$.

If L is a set of words, then $V(L)$ = the *variety* for L = the class of all groups satisfying every law in L.

Call a group G *uniformly Følner*, or *uniformly amenable* (UA), if in the definition of the Følner condition $\|E\|$ can be chosen to depend only on $\|A\|$ and ϵ, that is, if there is a function $F : \mathbb{N} \times (0,1) \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N}\ \forall A \subseteq G \text{ s.t. } \|A\| < n\ \forall \epsilon > 0,$$
$$\exists E \subseteq G \text{ s.t. } \|E\| < F(n,\epsilon)\ \&\ \forall a \in A\ \frac{\|E \triangle aE\|}{\|E\|} < \epsilon$$

A *class* $\mathcal{D}$ of groups is amenable if every group in $\mathcal{D}$ is amenable. A class $\mathcal{D}$ of groups is *uniformly amenable* if there is a single function $F : \mathbb{N} \times (0,1) \to \mathbb{N}$ that witnesses UA for all the groups in $\mathcal{D}$.

These are standard definitions, but the concept is actually the natural one from nonstandard analysis discussed above:

Theorem 5.1. *Let G be a group. TFAE: (1) G is UA; (2) ${}^*G$ is UA; (3) ${}^*G$ is amenable.*

Proof. $(1 \Rightarrow 2)$ Let F witness UA of G. Claim: F witnesses UA of ${}^*G$ as well. Let n, ϵ be given, and let $A \subseteq {}^*G$ with $\|A\| < n$. By transfer, ${}^*F : {}^*\mathbb{N} \times {}^*(0,1) \to {}^*\mathbb{N}$ witnesses *UA, so

$$\exists E \in {}^*\mathcal{P}(G),\ {}^*\|E\| \leq {}^*F(n,\epsilon)\ \&\ \forall a \in A\ \frac{{}^*\|E \triangle aE\|}{{}^*\|E\|} < \epsilon.$$

Note that an internal subset E of ${}^*G$ which has internal cardinality $\leq^* F(n,\epsilon)$ is externally finite with an actual, standard finite cardinality less than $F(n,\epsilon)$, since n and r are standard and ${}^*F(n,\epsilon) = F(n,\epsilon)$. This proves the claim.

$(2 \Rightarrow 3)$ is trivial.

$(3 \Rightarrow 1)$ Let $n \in \mathbb{N}, \epsilon > 0$ be given. We need to define $F(n,\epsilon)$. Let $m \in {}^*\mathbb{N} \setminus \mathbb{N}$. By amenability of ${}^*G$ and the Følner condition,

$$\forall A \in \mathcal{P}(G) \|A\| < n \Rightarrow \exists E \in \mathcal{P}(G),\ \|E\| \text{ finite}\ \&\ \forall a \in A\ \frac{\|E \cap aE\|}{\|E\|} < \epsilon.$$

Since any subset of ${}^*G$ with (standard) finite cardinality is internal, and any finite set has cardinality less than m, it follows that

$$\exists m \in {}^*\mathbb{N} \forall A \in {}^*\mathcal{P}(G) \|A\| < n \Rightarrow \exists E \in {}^*\mathcal{P}(G),\ {}^*\|E\| \leq m\ \&\ \forall a \in A\ \frac{{}^*\|E \triangle aE\|}{{}^*\|E\|} < \epsilon.$$

By transfer, there is a standard finite m that works for this n and r; put $F(n,\epsilon) := m$.

⊣

Corollary 5.1. *A subgroup or homeomorphic image of a UA group is UA.*

Proof. If H is a subgroup of G, and G is UA, then ${}^*G$ is amenable, ${}^*H$ must (as an external group) be amenable, so H is UA. A similar argument works for homeomorphic images (since the homeomorphic image of an amenable group is amenable). ⊣

Theorem 5.2. *Let $\mathcal{G}$ be a set of groups; then $\mathcal{G}$ is uniformly amenable iff ${}^*\mathcal{G}$ is amenable.*

Proof. $(\Rightarrow)$ If F witnesses UA for $\mathcal{G}$ then it witnesses amenability for every $G \in {}^*\mathcal{G}$ as in the proof of the last theorem.

(⇐) Fix $n \in \mathbb{N}, \epsilon > 0$ be given. We need to define $F(n,\epsilon)$. and $m \in {}^*\mathbb{N} \setminus \mathbb{N}$. By amenability of ${}^*\mathcal{G}$ and the Følner condition, m witnesses

$$\exists m \in {}^*\mathbb{N}\ \forall G \in {}^*\mathcal{G}\ \ \forall A \in {}^*\mathcal{P}(G)^*\|A\| < n \Rightarrow$$

$$\exists E \in {}^*\mathcal{P}(G),\ {}^*\|E\| \leq m\ \&\ \forall a \in A\ \frac{{}^*\|E \triangle aE\|}{{}^*\|E\|} < \epsilon$$

as above. By transfer, there is a standard finite m that works for this n and ϵ; put $F(n,\epsilon) := m$. ⊣

Proposition 5.1. *If V is a variety of groups and $\mathcal{G} \subseteq V$ then ${}^*\mathcal{G} \subseteq V$*

Proof. Let $\ell : w(x_1, \ldots, x_n)$ be a law for V, that is, $\ell \in L$ where $V = V(L)$. Now,

$$\forall G \in \mathcal{G}\ \ \forall g_1, \ldots, g_n \in G\ [w(g_1, \ldots, g_n) = e]$$

so by transfer,

$$\forall G \in^* \mathcal{G}\ \ \forall g_1, \ldots, g_n \in G\ [w(g_1, \ldots, g_n) = e]$$

so every $G \in^* \mathcal{G}$ satisfies ℓ. Since ℓ was arbitrary in L, ${}^*\mathcal{G} \subseteq V$. ⊣

Corollary 5.2. *Let V be a variety of groups. Then V is UA iff V is amenable.*

Proof. (⇒) is trivial. (⇐) Let $\mathcal{F} = {}^{\mathbb{N}\times(0,1)}\mathbb{N}$ be the set of all functions from $\mathbb{N} \times (0,1)$ to $\mathbb{N}$. Suppose V is *not* UA, then for every $F \in \mathcal{F}$ there is a $G_F \in V$ such that F does *not* witness UA for G_F. Let $\mathcal{G} = \{G_F\}_{F \in \mathcal{F}}$. Clearly $\mathcal{G}$ is not UA, so ${}^*\mathcal{G}$ is not amenable. But ${}^*\mathcal{G} \subseteq V$ by the last proposition, so V is not amenable. ⊣

6. Measures on topological spaces

Some of the most interesting nonstandard arguments in measure theory have involved the construction of measures on topological spaces. In this section I'll give a couple of examples which I think illustrate some of the types of arguments involved.

Suppose that X is a reasonable topological space (usually locally compact Hausdorff, but much of this machinery has also been developed for other nice spaces, such as Polish spaces). Denote the smallest $\sigma-$ algebra containing all open sets by $\mathcal{B}_X$ (or simply $\mathcal{B}$), for the *Borel* sets. A *Borel measure* is a measure on $(X, \mathcal{B}_X)$. A *Radon measure* is a Borel measure for which compact sets have finite measure and which is *compact-inner regular*: for every measurable $E \in \mathcal{B}_X$ and $\epsilon > 0$, there is a compact K and open U with $K \subseteq E \subseteq U$ and $\mu(U \setminus K) < \epsilon$.

Loeb measures are used to build Radon measures on spaces X by "pushing them down" using the standard part map. From general measure theory, we know the following change-of-variables theorem:

Lemma 6.1. *Let $(\Omega, \mathcal{A}, \nu)$ be a measure space, let $\mathcal{B}$ be a $\sigma-$algebra on a set X, and let $\phi : \Omega \to X$ be* measurable *in the sense that $\phi^{-1}(E) \in \mathcal{A}$ whenever $E \in \mathcal{B}$. Then $\mu(E) = \nu(\phi^{-1}(E))$ defines a measure on $(X, \mathcal{B})$.*

Exercise 6.1. *Prove this. Extend it to partial functions ϕ when the domain of ϕ is in $\mathcal{A}$.*

Call μ the *image* of ν under ϕ. Answering a question of economist Donald Brown, Robert Anderson[1] showed that every Radon measure is the image of a Loeb measure:

Theorem 6.1. *If $(X, \mathcal{B}, \mu)$ is a Radon measure space then μ is the image of μ_L under the standard part map, where $({}^*X, \mathcal{B}_L, \mu_L)$ is the (complete) standard measure space created from $({}^*X, {}^*\mathcal{B}, {}^{\circ *}\nu)$ using the Loeb construction.*

This already can be used to give an easy proof of a nontrivial theorem. Suppose that G is a *locally compact topological group*; that is, G is a group, it is also a locally compact topological space, and the group operations are continuous. We will see below that every such group has a *Haar measure*, that is, a Radon measure such that $\mu(E) = \mu(gE)$ for every $g \in G$. Lebesgue measure, for example, is Haar measure on the line (treating $\mathbb{R}$ as an additive group).

The following generalizes the classic analysis homework problem that if A is a Lebesgue measurable set of positive measure, then $A - A = \{a - b : a, b \in A\}$ contains an interval.

Theorem 6.2 (Steinhaus)**.** *Let $\langle G, e\rangle$ be a locally compact group with Haar measure μ, and let $A \subseteq G$ be a measurable set with $e \in A$ and $0 < \mu(A)$. Then AA^{-1} contains an open neighborhood of e.*

Proof. Without loss of generality A is compact; in particular, $\mu(A) < \infty$ and ${}^*A \subseteq \mathrm{st}^{-1} A$. Recall that ${}^{\circ *}\mu$ extends to a measure μ_L on ${}^*G$ such that for any measurable $B \subseteq G$, $\mathrm{st}^{-1} B$ is μ_L-measurable with $\mu(B) = \mu_L({}^*B) = \mu_L(\mathrm{st}^{-1} B)$.

Let $v \in monad(e), a \in {}^*A$; then ${}^\circ(va) = {}^\circ v {}^\circ a = {}^\circ a \in A$ (since A is compact); it follows that $v^*A \subseteq \mathrm{st}^{-1} A$. Since ${}^*A$ and v^*A are subsets of $\mathrm{st}^{-1} A$, and all three sets have the same nonzero μ_L-measure, ${}^*A$ and v^*A must intersect, that is, $a = vb$ for some $a, b \in {}^*A$. Then $v = ab^{-1}$. Since v is arbitrary, $monad(e) \subseteq {}^*(AA^{-1})$. The result follows. ⊣

Exercise 6.2. *Identify where we used Anderson's theorem in this proof.*

More generally, one wants to construct a standard measure on a topological space X by starting with a measure on some $\Omega \subseteq^* X$ and pushing it down using the standard part map. (Often the measure on Ω will be built using methods from finite combinatorics on a hyperfinite Ω, but this is by no means required.) In order to make this work, we need to know that $\mathrm{st} : NS(X) \to X$ is a measurable partial function. Unfortunately, this is not universally true. However, in the most common situations it will be. For example, if X is compact Hausdorff, $(\Omega, \mathcal{A}, \mu)$ is an internal *-fa *-measure with $\Omega \subseteq {}^*X$ hyperfinite and $\mathcal{A}$ the internal power set of Ω, then the restriction of st to Ω will be measurable from the complete Loeb space to X.

I'll end with an application of this technique.

Theorem 6.3. *Let G be a compact Hausdorff space, then G has a Haar measure.*

The same proof works in general with some modification on locally compact Hausdorff groups; the hardest part is confirming measurability of the standard part map.

Proof. By saturation there exists an internal ${}^*-$neighborhood u of e such that $e \in u \subseteq monad(e)$. $u^{{}^*G} = \{gu : g \in {}^*G\}$ is an internal open cover of the ${}^*-$compact set ${}^*G$, so there is a hyperfinite subcover u^Ω. Choose $\Omega = \{\omega_1, \dots, \omega_H\}$ so that H is ${}^*-$minimal. Let ν be the uniform ${}^*-$probability measure on $(\Omega, \mathcal{P}(\Omega))$. As noted above, the restriction st_Ω of the standard part map to Ω is Borel measurable. Let μ be the corresponding Borel image measure (under st_Ω) on $(G, \mathcal{B}_G)$.

Clearly μ is a probability measure; it remains to show that μ is invariant. Pick $g \in G$ and $B \in \mathcal{B}$. Let A be any internal subset of $\mathrm{st}^{-1} B$, and let $C = \bigcup_{a \in A} \{\omega \in \Omega : \omega u \cap g^{-1}(au) \neq \emptyset\}$. Exercise: show that (i) $C \subseteq \mathrm{st}^{-1}(g^{-1}B)$ and (ii) $\|C\| \geq \|A\|$. (Hint: (ii) uses minimality of H) This implies that $\mu(g^{-1}B) \geq \mu(B)$. Since g and B were arbitrary, μ must be invariant. ⊣

References

[1] Robert M. Anderson. Star-finite representations of measure spaces. *Trans. Amer. Math. Soc.*, 271(2):667–687, 1982.

[2] Leif O. Arkeryd, Nigel J. Cutland, and C. Ward Henson, editors. *Nonstandard analysis*, volume 493 of *NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences*. Kluwer Academic Publishers Group, Dordrecht, 1997. Theory and applications.

[3] Martin Davis. *Applied nonstandard analysis*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977. Pure and Applied Mathematics.

[4] C.W. Henson. Følner condition $\Longrightarrow$ amenability. *Trans. Amer. Math. Soc.*, 172:437–446, 1972.

[5] David A. Ross. Loeb measure and probability. In *Nonstandard analysis (Edinburgh, 1996)*, volume 493 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 91–120. Kluwer Acad. Publ., Dordrecht, 1997.

Department of Mathematics, University of Hawaii, Honolulu, HI 96822

E-mail address: `ross@math.hawaii.edu`

URL: `www.math.hawaii.edu/~ross`