Exercises 1

1. Prove that in *any* ordered field proper extension of $\mathbb{R}$ there are infinitesimals. (An ordered field is just a field with a linear order $\leq$ which respects the field operations in the usual way, just like with $\mathbb{R}$ and $\mathbb{Q}$.) In other words, we don't really need things like transfer and saturation to get infinitesimals, just an ordered field strictly extending $\mathbb{R}$.

2. Prove that we could have defined $\mathbb{V}_n(S)$ by $\mathbb{V}_0(S) := S; \quad \mathbb{V}_{n+1}(S) = S \cup \mathcal{P}(\mathbb{V}_n)$.

The next three problems – proving properties stated in the lecture notes – are straightforward, you shouldn't need any saturation.

3. Prove this property of ${}^*\mathbb{R}$: If x and y are finite then $x + y$ is finite.

4. Prove this property of ${}^*\mathbb{R}$: If α is infinite then $\alpha^{-1} \approx 0$.

5. Prove this property of ${}^*\mathbb{R}$: If $x \approx y$ and $u \approx v$ and x and u are finite then $xu \approx yv$

For the next two you will need some saturation!

6. If $A, B \subseteq \mathbb{R}$ and every element of A is less than every element of B then there exists some $z \in {}^*\mathbb{R}$ such that $a < z < b$ for *every* $a \in A, b \in B$.

7. If $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \dots$, where A_n is internal, then $\bigcup_{n \in \mathbb{N}} A_n$ is external.