

Exercises 3

1. Give a nonstandard proof that a closed and bounded subset K of \mathbb{R} is compact.
2. Suppose X, Y are Hausdorff, and Ω is “S-dense” in *X (that is, $\text{st}_X(\Omega) = X$ — equivalently, for every open subset u of X , ${}^*u \cap \Omega \neq \emptyset$). Suppose further that $F : \Omega \rightarrow {}^*Y$ is internal and satisfies the following:

$$F(NS(X) \cap \Omega) \subseteq NS(Y) \text{ and } (\forall x, y \in NS(X) \cap \Omega)[(x \approx y) \Rightarrow F(x) \approx F(y)]$$

(We call this *S-Continuity* of F .) Prove that $f(x) := {}^\circ F(\omega)$, where $\omega \approx x$, is well-defined and continuous on X .

3. Let (X, d) be a metric space, and let $PNS(X) := \{y \in {}^*X \mid (\forall \epsilon > 0)(\exists x \in X)(y \approx x)\}$ (the *prenearstandard* points of *X). Show that $PNS(X)/\approx$ is complete, and that if Y is any complete metric space into which X embeds isometrically then in fact $PNS(X)/\approx$ embeds into Y isometrically. (In other words, $PNS(X)/\approx$ is the completion of X .)
4. Exercise 2.4 in the Metric Spaces chapter