

# Nonstandard Analysis (Math 649k) Spring 2008

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February 29, 2008

# 1 Introduction

## 1.1 Outline of course:

1. Introduction: motivation, history, and propoganda
2. Nonstandard models: definition, properties, some unavoidable logic
3. Very basic Calculus/Analysis
4. Applications of saturation
5. General Topology
6. Measure Theory
7. Functional Analysis
8. Probability

## 1.2 Some References:

1. Abraham Robinson (1966) *Nonstandard Analysis* North Holland, Amsterdam
2. Martin Davis and Reuben Hersh *Nonstandard Analysis* Scientific American, June 1972
3. Davis, M. (1977) *Applied Nonstandard Analysis* Wiley, New York.
4. Sergio Albeverio, Jens Erik Fenstad, Raphael Høegh-Krohn, and Tom Lindstrøm (1986) *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*. Academic Press, New York.
5. Al Hurd and Peter Loeb (1985) *An introduction to Nonstandard Real Analysis* Academic Press, New York.
6. Keith Stroyan and Wilhelminus Luxemburg (1976) *Introduction to the Theory of Infinitesimals* Academic Press, New York.
7. Keith Stroyan and Jose Bayod (1986) *Foundations of Infinitesimal Stochastic Analysis* North Holland, Amsterdam
8. Leif Arkeryd, Nigel Cutland, C. Ward Henson (eds) (1997) *Nonstandard Analysis: Theory and Applications*, Kluwer
9. Rob Goldblatt (1998) *Lectures on the Hyperreals*, Springer

### 1.3 Some history:

- (xxxx) Archimedes
- (1615) Kepler, *Nova stereometria solidorum vinariorum*
- (1635) Cavalieri, *Geometria indivisibilis*
- (1635) *Excercitationes geometricae* ("Rigor is the affair of philosophy rather than mathematics")
- (1684-6) Leibniz, *Acta Eruditorum*
- (1691) Newton, *De Quadratura Curvarum* (disavows infinitesimals)
- (1691) Johann Bernoulli
- (>1700) Leibniz again (the two questions: (a) are infinitesimals real, and (b) do infinitesimals lead to correct theorems, are independent)
- (1734) Berkeley, *The analyst* ("ghosts of departed quantities")
- (1748–1770) Euler, *Introductio in analysin infinitorum* (etc.)
- (18xx) Bolzano (IVT)
- (1821) Cauchy, *Cours d'analyse* (also IVT)
- (1822) *Resume des lecons sur le calcul infinitesimal* (introduces his notion of 'infinitesimal')
- (1872) Dedekind and Cantor define the continuum.
- (1887/8) Cantor (claims to prove impossibility of infinitely small numbers)
- (1934) Skolem (nonstandard models of the integers)
- (1948) E. Hewitt (hyperreal fields)
- (1955) Los (ultrapowers)
- (1958) Laugwitz and Schmeiden, *Eine Erweiterung der Infinitesimalrechnung*
- (1961,1966) A. Robinson
- (1970s) Nonstandard hull constructions (Luxemburg; Loeb)
- (1976) Keisler's *Calculus* text

## 1.4 Some properties of $\mathbb{R}$ :

1.  $\mathbb{R}$  is a field.
2.  $\mathbb{R}$  is a *linearly ordered* field
3.  $\mathbb{R}$  is Archimedean
4.  $\mathbb{R}$  is Real-closed
5.  $\mathbb{R}$  is (order-) complete

## Naive/alternate ways to extend $\mathbb{R}$ :

1. Use any nonarchimedean field of ch. 0, eg (P) series
2. Add an infinitesimal and write down some axioms (Keisler et al)
3. Compactness theorem from Logic (Skolem)
4. Surreal numbers
5. Syntactic, eg  $\exists^{st} x P(x)$  (Nelson's IST)
6.  $\neg$ “There are no infinitesimals.” (“Smooth infinitesimals” - Bell et al, from Lawvere)

## Problems with these:

1. What about functions? (EG:  $\sin(x + \delta x)$ )
2. Want to prove ‘real’ theorems.
3. Want to prove ‘new’ theorems.

## 2 Universes

To define a *nonstandard universe*...

### 2.1 Standard universe

The standard universe should:

1. Contain all the mathematical objects we will need:

- $0, 1, 2, \dots \in \mathbb{V}$  (i.e.,  $\mathbb{N} \subseteq \mathbb{V}$ )
- More generally  $\mathbb{R} \subseteq \mathbb{V}$
- $\mathbb{N}, \mathbb{R} \in \mathbb{V}$
- ${}^{\mathbb{R}}\mathbb{R} \in \mathbb{V}$ ,  ${}^{\mathbb{R}}\mathbb{R} \subseteq \mathbb{V}$ , etc.

2. Satisfy reasonable closure properties, eg

- (a) If  $A \in \mathbb{V}$  then  $\mathcal{P}(A) \in \mathbb{V}$
- (b) (Transitivity) If  $A \in \mathbb{V}$  then  $A \subseteq \mathbb{V}$
- (c)  $\mathbb{V}$  closed under  $\cap, \cup, \setminus$

Remark: From basic set theory we know that if these closure properties already give us all of the familiar mathematical operations. For example, if  $A \in \mathbb{V}$  (and so  $A \subseteq \mathbb{V}$ ) then:

1. If  $a, b \in A$  then  $(a, b) \in \mathbb{V}$  (since  $(a, b) = \{\{a\}, \{a, b\}\} \in \mathcal{P} \mathcal{P} A$ )
2.  $A^2, {}^A A, {}^A \mathbb{R} \in \mathbb{V}$  (assuming  $\mathbb{R} \in \mathbb{V}$ )
3. The function  $f \rightarrow \int f dx$  (from  $C(\mathbb{R})$  to  $C(\mathbb{R})$ ) is in  $\mathbb{V}$
4. etc.

One convenient way to build such a universe is as a *superstructure*:

**Definition 2.1.** Let  $S$  be a set of urelements (or individuals), that is, objects assumed not to be sets themselves. Define sets  $V_n(S)$  inductively by  $V_0(S) := S$ ;  $V_{n+1}(S) := V_n(S) \cup \mathcal{P}(V_n(S))$ . The set  $V(S) := \bigcup_{n=0}^{\infty} V_n(S)$  is the superstructure over  $S$ .

Remark:

1. When there is no room for confusion write  $V_n$  and  $V$  instead of  $V_n(S)$  and  $V(S)$ .

2. In set theory it is common to define the natural numbers by  $0 := \{\}$ ;  $n + 1 := n \cup n$  (that is,  $n = \{0, 1, \dots, n - 1\}$ ). It is easy to see that with this definition  $n \in V_{n+1}$  (even if we start with  $S = \emptyset$ ). Similarly, can define  $\mathbb{Z}$  and  $\mathbb{Q}$  in such a way that  $\mathbb{Q} \subseteq V(S)$ . However, if  $S$  is finite or countable then  $V(S)$  is countable (SHOW!), in which case  $\mathbb{R} \not\subseteq V$
3. Again, in set theory one can get  $\mathbb{R}$  into our universe (with  $S = \emptyset$ ) by letting the index  $n$  on  $V_n$  range over a bigger ordinal than  $\omega$  ( $\omega + \omega$  works). Instead of doing that, we will *ASSUME THAT*  $\mathbb{R} \subseteq S$ . In particular, we will *not* assume that the elements of  $\mathbb{R}$  are Dedekind cuts (or any other kind of set).

**Theorem 2.1.** *Let  $S, V = V(S)$  be as above. Then:*

1.  $V_n \in V$ .
2. (Transitivity) If  $x \in A \in V$  then  $x \in V$ . If  $x \in A \in V_n$  then  $x \in V_{n-1}$ .
3. If  $A \in (V \setminus S)$  then  $\mathcal{P}(A) \in V$ . If  $A \in (V_n \setminus S)$  then  $\mathcal{P}(A) \in V_{n+2}$
4. If  $A \subseteq B \in V$  then  $A \in V$ .
5.  $A, B \in (V \setminus S)$  then  $A \cap B, A \cup B, (A \setminus B) \in (V \setminus S)$
6. If  $A_1, \dots, A_n \in (V \setminus S)$  then  $(A_1 \times \dots \times A_n) \in V$
7. If  $A, B \in (V \setminus S)$  and  $f : A \rightarrow B$  then  $f \in V$
8. If  $I, X \in V$  and  $A_i \in X$  for every  $i \in I$  then  $\bigcup_{i \in I} A_i \in V$  and  $\prod_{i \in I} A_i \in V$
9. Other properties will be added here as needed.

Some proofs will come later.

## 2.2 Nonstandard universe

Suppose  $V$  is a standard universe (not necessarily a superstructure). A *nonstandard universe* will consist of a proper extension  ${}^*V$ , and an injection  $*$ :  $V \rightarrow {}^*V$  that satisfies certain properties. Roughly, these properties are:

1. Transfer: every *first order* mathematical statement true in  $V$  is true in  ${}^*V$  and vice versa; and
2. Saturation: Any ‘sufficiently small’ infinite set of first-order conditions on a mathematical object which is finitely satisfiable in  ${}^*V$  is satisfiable in  ${}^*V$

To make these precise, we need 2 ideas:

1. Definition of “first-order statement/condition”
2. Notion of “Internal/External” set (for formulating saturation)

## 2.3 Logic

Fix a standard universe  $\mathbb{V}$ ; we’ll define a *first-order language*  $\mathcal{L}_{\mathbb{V}}$  as follows:

### 2.3.1 Alphabet for all languages:

- Logical connectives:  $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- Quantifier symbols:  $\forall, \exists$
- Variables:  $x, y, z, \dots; x_0, x_1, x_2, \dots; \epsilon, \delta, \dots$
- Punctuation:  $), ($
- A binary relation symbol  $\in$

### 2.3.2 Symbols for a fixed standard universe $\mathbb{V}$ :

- A *constant symbol*  $c$  for every constant  $c$  in  $\mathbb{V}$ ;
- An  $n$ -ary *function symbol*  $f$  for every  $n$ -ary function  $f$  in  $\mathbb{V}$ ;
- An  $n$ -ary *predicate symbol*  $P$  for every  $n$ -ary predicate (relation)  $P$  in  $\mathbb{V}$ ;

#### Remarks:

1. For simplicity we are using the same symbols both as elements of  $\mathbb{V}$  and as formal symbols in the language. Hopefully this will cause no confusion.

2. Some elements of  $\mathbb{V}$  might play dual roles; for example,  $\mathbb{R}$  makes sense as both a constant (for the object which is set of reals) and as the unary relation “is a real number”. This makes no difference, as equivalent statements such as “ $x \in \mathbb{R}$ ” and “ $\mathbb{R}(x)$ ” will always both be true or both false when appropriately interpreted.
3. We could simply treat everything as an object/set, and rely on the  $\in$  symbol for all representations. For example, instead of formalizing an  $n$ -ary function  $f$  with a function symbol, we could simply always write  $(x_1, \dots, x_n) \in f$  instead of  $f(x_1, \dots, x_n)$ . Alternately, we could reserve constant symbols for urelements. IT DOESN'T MATTER!
4. The reason we don't include  $\in$  among the other binary relation symbols is that we always want to interpret it with normal “element of”, instead of some fishy nonstandard interpretation.

To define  $\phi$  is a formula of  $\mathcal{L}_{\mathbb{V}}$ :

- Definition of ‘term’
  - If  $c$  is a constant symbol then  $c$  is a term
  - If  $x$  is a variable then  $x$  is a term
  - If  $f$  is an  $n$ -ary function symbol and  $\tau_1, \dots, \tau_n$  are terms then  $f(\tau_1, \dots, \tau_n)$  is a term
- Definition of ‘atomic formula’
  - If  $\tau_1$  and  $\tau_2$  are terms then  $\tau_1 = \tau_2$  and  $\tau_1 \in \tau_2$  are atomic formulas
  - If  $P$  is an  $n$ -ary predicate symbol and  $\tau_1, \dots, \tau_n$  are terms then  $P(\tau_1, \dots, \tau_n)$  is an atomic formula
- Definition of ‘formula’
  - Every atomic formula is a formula
  - If  $\phi, \psi$  are formulas then so are  $(\neg\phi), (\phi \vee \psi), (\phi \wedge \psi), (\phi \rightarrow \psi)$ , and  $(\phi \leftrightarrow \psi)$
  - If  $\phi$  is a formula,  $x$  a variable symbol, and  $\tau$  a term, then  $(\forall x \in \tau)\phi$  and  $(\exists x \in \tau)\phi$  are formulas

**Remarks:**

1. Terms are essentially constants, variables, or functions.
2. Formulas are statements whose “value” is a truth-value.
3. It is usually convenient to take certain logical connectives as primitive and define the rest in terms of them, for example start with  $\vee$  and  $\neg$  and define  $(\phi \wedge \psi)$  by  $(\neg((\neg\psi) \vee (\neg\phi)))$

4. Similarly, the existential quantifier  $\exists$  can be taken as primitive, and  $\forall$  defined by  $(\forall x \in \tau)\phi := (\neg(\exists x \in \tau)(\neg\phi))$
5. Note the use of *bounded quantifiers*  $\forall x \in \tau$  and  $\exists x \in \tau$ , instead of the more common  $\forall x$  and  $\exists x$ . This is a technical convenience, and instead we could have just used ‘ordinary’ quantifiers and later restricted ourselves (in the statement of the Transfer Theorem) to a subset of formulas.
6. Parentheses will be employed in a haphazard manner when there is no room for confusion.

This inductive definition of “formula” means that anything else we do with formulas (transformations, theorems, etc.) will be done by induction. For example, here’s a definition of *sentence*, which is a formula with no variables “free”:

**Definition 2.2.** Define  $FV(\phi)$ ,  $\phi$  a term or formula, by:

1.  $FV(c) := \emptyset$ ,  $c$  a constant symbol
2.  $FV(x) := \{x\}$ ,  $x$  a variable
3.  $FV(f(\tau_1, \dots, \tau_n)) := \bigcup_{i=1}^n FV(\tau_i)$ , if  $f$  is an  $n$ -ary function symbol and  $\tau_1, \dots, \tau_n$  are terms
4.  $FV(\tau_i = \tau_j) := FV(\tau_i) \cup FV(\tau_j)$ ,  $\tau_i, \tau_j$  terms
5.  $FV(\tau_i \in \tau_j) := FV(\tau_i) \cup FV(\tau_j)$ ,  $\tau_i, \tau_j$  terms
6.  $FV(P(\tau_1, \dots, \tau_n)) := \bigcup_{i=1}^n FV(\tau_i)$ , if  $P$  is an  $n$ -ary predicate symbol and  $\tau_1, \dots, \tau_n$  are terms
7. If  $\phi$  and  $\psi$  are formulas then  $FV(\neg\phi) := FV(\phi)$ ,  $FV(\phi \vee \psi) := FV(\phi) \cup FV(\psi)$  (etc.)
8. If  $\phi$  is a formula,  $x$  a variable, and  $\tau$  is a term, then  $FV((\exists x \in \tau)\phi) := (FV(\phi) \cup FV(\tau)) \setminus \{x\}$

A sentence is a formula  $\phi$  with  $FV(\phi) = \emptyset$

We want to define what it means for a formula  $\phi$  to be “true in  $\mathbb{V}$ ”. However, there is a technical difficulty, namely, that if  $\phi$  is not a sentence then it might not have just one truth value. Instead, we define something more general.

Start with terms. Suppose  $\tau = \tau(x_1, \dots, x_n)$  is a term with  $FV(\tau) \subseteq \{x_1, \dots, x_n\}$ . Let  $\vec{a} = (a_1, \dots, a_n) \in \mathbb{V}^n$ , define  $\tau[\vec{a}]$  (the interpretation of  $\tau$  in  $\mathbb{V}$  with parameters  $\vec{a}$ ) inductively by:

- $c[\vec{a}] := c$ ;
- $x_i[\vec{a}] := a_i, \quad i \leq n$ ;
- $f(\tau_1, \dots, \tau_k)[\vec{a}] := f(\tau_1[\vec{a}], \dots, \tau_k[\vec{a}])$

Remarks:

1. Note that  $\tau[\vec{a}]$  is an element of  $\mathbb{V}$
2. Strictly speaking, the above definition depends on  $n$  (and some given enumeration of the variables in the language). However, (EXERCISE) a simple inductive argument shows that if also  $FV(\tau) \subseteq \{y_1, \dots, y_m\}$  and  $\vec{b} \in \mathbb{V}^m$  and  $a_i = b_j$  whenever  $x_i = y_j$  then  $\tau[\vec{a}] = \tau[\vec{b}]$

We now do something similar to interpret formulas in  $\mathbb{V}$ . Suppose  $\phi = \phi(x_1, \dots, x_n)$  is a formula with  $FV(\phi) \subseteq \{x_1, \dots, x_n\}$ . Let  $\vec{a} = (a_1, \dots, a_n) \in \mathbb{V}^n$ , define  $\mathbb{V} \models \phi[\vec{a}]$  ( $\phi$  is true in  $\mathbb{V}$  with parameters  $\vec{a}$ ) inductively by:

- $\mathbb{V} \models (\tau_1 = \tau_2)[\vec{a}]$  provided  $\tau_1[\vec{a}] = \tau_2[\vec{a}]$
- $\mathbb{V} \models (\tau_1 \in \tau_2)[\vec{a}]$  provided  $\tau_1[\vec{a}] \in \tau_2[\vec{a}]$
- $\mathbb{V} \models P(\tau_1, \dots, \tau_k)[\vec{a}]$  provided  $(\tau_1[\vec{a}], \dots, \tau_k[\vec{a}]) \in P$
- $\mathbb{V} \models (\neg\phi)[\vec{a}]$  provided  $\mathbb{V} \not\models \phi[\vec{a}]$
- $\mathbb{V} \models (\phi \vee \psi)[\vec{a}]$  provided  $\mathbb{V} \models \phi[\vec{a}]$  or  $\mathbb{V} \models \psi[\vec{a}]$
- $\mathbb{V} \models (\exists x \in \tau)\phi[\vec{a}]$  provided for some  $b \in \tau[\vec{a}]$ ,  $\mathbb{V} \models \phi[\vec{a}; b/x]$  (where  $\phi[\vec{a}; b/x]$  means that if  $x$  is free for  $\phi$ , i.e.,  $x = x_i$ , then  $\phi[\vec{a}; b/x] := \phi[a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n]$ )

## 2.4 Fundamental Theorem

Now, suppose  $*$  is any injection from  $\mathbb{V}(S)$  into  $\mathbb{V}(T)$ . Write  $*a$  instead of  $*(a)$  for any  $a \in \mathbb{V}(S)$ . We can transform any term or formula  $\phi$  of  $\mathcal{L}_{\mathbb{V}(S)}$  into a term or formula  $*\phi$  of  $\mathcal{L}_{\mathbb{V}(T)}$  by replacing every symbol  $\sigma$  appearing in  $\phi$  which comes from  $\mathbb{V}(S)$  by the symbol  $*\sigma$  from  $\mathbb{V}(T)$ . Formally:

1.  $*c := c$ ,  $c$  a constant symbol
2.  $*x := x$ ,  $x$  a variable symbol
3.  $*(f(\tau_1, \dots, \tau_n)) := *f(*\tau_1, \dots, *\tau_n)$
4.  $*(\tau_1 = \tau_2) := (*\tau_1 = *\tau_2)$
5.  $*(\tau_1 \in \tau_2) := (*\tau_1 \in *\tau_2)$
6.  $*(P(\tau_1, \dots, \tau_n)) := *P(*\tau_1, \dots, *\tau_n)$
7.  $*(\neg\phi) := \neg*\phi$
8.  $*(\phi \vee \psi) := (*\phi \vee *\psi)$
9.  $*(\exists x \in \tau)\phi := (\exists x \in *\tau)*\phi$

Now, relative to this injection  $*$  call a subset  $E \subseteq \mathbb{V}(T)$ :

- *Standard* if  $E = *A$  for some  $A \in \mathbb{V}(S)$
- *Internal* if  $E \in *A$  for some  $A \in \mathbb{V}(S)$
- *External* otherwise

The fundamental theorem of nonstandard analysis can now be stated.

**Theorem 2.2.** *Let  $\mathbb{V}(S)$  be any superstructure, and  $\kappa$  an infinite cardinal. There is then a superstructure  $\mathbb{V}(T)$ , and an injection  $*$  from  $\mathbb{V}(S)$  into  $\mathbb{V}(T)$ , such that  $T = *S$ , satisfying:*

0. *If  $A \in \mathbb{V}(S)$  is infinite then  $*A \neq \{*a \mid a \in A\}$*
1. *(Transfer) If  $\phi$  is a sentence of  $\mathcal{L}_{\mathbb{V}(S)}$ , then*

$$\mathbb{V}(S) \models \phi \quad \text{if and only if} \quad \mathbb{V}(*S) \models *\phi$$

2. *( $\kappa$ -saturation) If  $\mathcal{A}$  is a (possibly external) collection of internal subsets of  $\mathbb{V}(*S)$ ,  $\text{card}(\mathcal{A}) < \kappa$ , and  $\mathcal{A}$  satisfies the finite intersection property, then  $\bigcap \mathcal{A} \neq \emptyset$*

Recall that a collection  $\mathcal{A}$  satisfies the *finite intersection property* (FIP) provided for every finite  $\mathcal{A}' \subset \mathcal{A}$ ,  $\bigcap \mathcal{A}' \neq \emptyset$ .

For example, if  $\kappa = \aleph_1$ , condition (2) says that if  $\{A_n\}_{n=0}^{\infty}$  is a sequence of internal sets and  $\bigcap_{n=0}^{\infty} A_n = \emptyset$  then  $\bigcap_{n=0}^N A_n = \emptyset$  for some finite  $N$ .

**Exercise 2.1.** Express each of the following as a first-order formula in  $\mathcal{L}_{\forall}$  without referring to any constants of  $\mathbb{V} = \mathbb{V}(S)$ . You may use earlier problems in your solutions of later ones. I've done two of them for you.

1.  $\phi(x) = \text{"}x \text{ is a set"}$
2.  $\phi(x, y) = \text{"}x \subseteq y \text{"}$
3.  $\phi(x, y_1, \dots, y_n) = \text{"}x = \{y_1, \dots, y_n\} \text{"}$
4.  $\phi(x, y) = \text{"}z = (x, y) \text{"}$
5.  $\phi(X, Y, Z) = \text{"}Z = X \times Y \text{"}$
6.  $\phi(f, X, Y) = \text{"}f \text{ is a function from } X \text{ into } Y \text{"}$
7.  $\phi(f, X, Y) = \text{"}f \text{ is a function from } X \text{ onto } Y \text{"}$

Some solutions:

" $x$  is a set":  $(\exists y \in x)(y = y)$  (Note the other obvious answer -  $\neg(x \in S)$  - is not allowed because it refers to the constant  $S$  of  $\mathbb{V}(S)$ )

" $z = (x, y)$ ":  $(\exists u \in z)(\exists v \in z) \text{"}u = \{x\} \text{"} \wedge \text{"}v = \{x, y\} \text{"} \wedge \text{"}z = \{u, v\} \text{"}$  (Note that I've assumed that  $\phi(x, y_1, \dots, y_n) = \text{"}x = \{y_1, \dots, y_n\} \text{"}$  is first-order)

## 2.5 Some consequences of Transfer

**Lemma 2.1.** *Suppose  $\mathbb{V}(S), \mathbb{V}(*S), *$  as above.*

1. *If  $a, A \in \mathbb{V}(S)$  then  $a \in A \iff *a \in *A$*
2. *If  $A, B \in \mathbb{V}(S)$  then  $A \subseteq B \iff *A \subseteq *B$*
3. *If  $A_1, A_2, \dots, A_n \in \mathbb{V}(S)$  then*  

$$*(A_1 \cap A_2 \cap \dots \cap A_n) = (*A_1 \cap *A_2 \cap \dots \cap *A_n)$$
4. *If  $A_1, A_2, \dots, A_n \in \mathbb{V}(S)$  then*  

$$*(A_1 \cup A_2 \cup \dots \cup A_n) = (*A_1 \cup *A_2 \cup \dots \cup *A_n)$$
5. *If  $A, B \in \mathbb{V}(S)$  then  $*(A \setminus B) = (*A) \setminus (*B)$ .*
6.  $*\emptyset = \emptyset$
7. *If  $A \in \mathbb{V}(S)$  then  $*\mathcal{P}(A) \subseteq \mathcal{P}(*A)$*
8.  $*\mathbb{V}_n(S) \subseteq \mathbb{V}_n(*S)$
9. *If  $a_1, a_2, \dots, a_n \in \mathbb{V}(S)$  then  $*\{a_1, a_2, \dots, a_n\} = \{*a_1, *a_2, \dots, *a_n\}$*
10. *If  $a_1, a_2, \dots, a_n \in \mathbb{V}(S)$  then  $*(a_1, a_2, \dots, a_n) = (*a_1, *a_2, \dots, *a_n)$*
11. *If  $A_1, A_2, \dots, A_n \in \mathbb{V}(S)$  then*  

$$*(A_1 \times A_2 \times \dots \times A_n) = (*A_1 \times *A_2 \times \dots \times *A_n)$$
12. *If  $F : A \rightarrow B$  is a function ( $F, A, B \in \mathbb{V}(S)$ ) then  $*F$  is a function from  $*A$  to  $*B$ .*
13. *If  $F : A \rightarrow B$  is a function,  $C \subseteq A$ , and  $F, A, B, C \in \mathbb{V}(S)$ , then*  

$$*(F \upharpoonright_C) = *F \upharpoonright_{*C}$$

Proof:

1. Apply transfer to the sentence  $a \in A$  (cf Exercise 2.1)
2. Apply transfer to the sentence  $\forall x \in A(x \in B)$
3. For some  $k$ ,  $A_1, A_2, \dots, A_n \in \mathbb{V}_k(S)$ . Now apply transfer to the sentence  

$$\forall x \in \mathbb{V}_k(S)(x \in (A_1 \cap A_2 \cap \dots \cap A_n) \iff (x \in A_1 \wedge \dots \wedge x \in A_n))$$
4. Exercise
5. Exercise
6.  $*\emptyset = *A \setminus A = *A \setminus *A = \emptyset$
7. Apply transfer to the sentence  $\forall y \in \mathcal{P}(A)(y \subseteq A)$
8. Induct on  $n$ , and use the last property.

9. Exercise (cf Exercise 2.1)
10. Immediate by the last property
11. Exercise (cf Exercise 2.1)
12. Exercise
13. Exercise

## 2.6 Internality

The above mainly concerns properties of standard sets. We now consider properties of *internal* sets. The importance of internal sets can be seen in the transfer principle: if a quantifier  $(\forall x \in A)$  appears in a sentence  $\phi$ , then the corresponding quantifier in  ${}^*\phi$  is  $(\forall x \in {}^*A)$ , that is, the quantifier ranges only over internal sets. That this is generally the case (if we replace  $A$  by a term  $\tau$ ) follows from the fact that terms are really functions, from 12 in the last Lemma, and from the following:

**Proposition 2.1.** *If  $C \in B$  and  $B$  internal then  $C$  is internal.*

Proof: For some  $A \in \mathbb{V}(S)$ ,  $B \in {}^*A$ . For some  $n$ ,  $A \in \mathbb{V}_n(S)$ . Now, apply transfer to the sentence  $(\forall x \in A)(\forall y \in x)(y \in \mathbb{V}_n(S))$ .

Implicit in this proof is the idea that for internality we shouldn't need to look at *all* standard sets, just the sets  $\mathbb{V}_n(S)$ . this is the first part of Lemma 2.2 below.

We know from the above that if  $A \in \mathbb{V}(S)$  then  $\{{}^*B \mid B \subseteq A\} \subseteq {}^*\mathcal{P}(A) \subseteq \mathcal{P}({}^*A)$ , and that the first subset is strict is and only if  $A$  is infinite. (The same can be shown for the second subset.) In fact,  ${}^*\mathcal{P}(A)$  is the set of internal subsets of  $A$ . However, we can do better than this.

Note that while the power set *function* is not an element of  $\mathbb{V}(S)$ , for any  $n$  the restriction  $\mathcal{P}_n$  of  $\mathcal{P}$  to  $\mathbb{V}_n(S)$  is a function from  $\mathbb{V}_n(S)$  to  $\mathbb{V}_{n+2}(S)$  (and so is an element of  $\mathbb{V}(S)$ ). It follows that  ${}^*\mathcal{P}_n$  is a function from  ${}^*\mathbb{V}_n(S)$  to  ${}^*\mathbb{V}_{n+2}(S)$ . By the result about restrictions of functions, we will abuse notation and write  ${}^*\mathcal{P}$  instead of  ${}^*\mathcal{P}_n$ .

**Proposition 2.2.** *For any internal  $A$ ,  ${}^*\mathcal{P}(A)$  is the (internal) set of internal subsets of  $A$ .*

Proof: As in Proposition 2.1, for some  $n$   $A \in {}^*\mathbb{V}_n(S)$ .

For  $\supseteq$ , apply transfer to  $(\forall y \in \mathbb{V}_n(S))(\forall x \in \mathbb{V}_n(S))(x \in \mathcal{P}_n(y) \iff "x \subseteq y")$

For  $\subseteq$ , if  $B \in {}^*\mathcal{P}(A)$  then  $B \in {}^*\mathbb{V}_{n+2}(S)$  (from Lemma 2.1), so is internal. Another application of transfer gives that  $B \subseteq A$ .

**Corollary 2.1.** *the set of infinitesimals in  ${}^*\mathbb{R}$  is either  $\{0\}$  or is external.*

Proof: Recall that this set is  $I := \bigcap_{n=1}^{\infty} {}^*(-\frac{1}{n}, \frac{1}{n})$ . Suppose  $\{0\} \subsetneq I$ . Exercise:  $(\forall x \in I)(2x \in I \wedge -x \in I)$ . If  $I$  is internal (hence in  ${}^*\mathcal{P}([-1, 1])$ ) then  $M = {}^*\sup(I)$  exists, moreover  $M > 0$ . If  $M \in I$  then  $2M \in I$  else  $M/2 \notin I$ , a contradiction either way.

**Lemma 2.2.** *Let  $E \subseteq \mathbb{V}({}^*S)$ ; the following are equivalent:*

1.  $E$  is internal
2.  $E \in {}^*\mathbb{V}_n(S)$  for some  $n \in \mathbb{N}$
3. There is a formula  $\phi(x, y_1, \dots, y_n)$  in  $\mathcal{L}_{\mathbb{V}(S)}$  ( $n \geq 0$ ), and internal sets  $B, b_1, \dots, b_n$ , such that

$$E = \{e \in B \mid \mathbb{V}({}^*S) \models {}^*\phi[e, b_1, \dots, b_n]\}$$

The equivalence of (1) and (2) means that the collection of internal sets is simply  $\bigcup_{n=0}^{\infty} {}^*\mathbb{V}_n(S)$ .

The equivalence of (1) and (3) is often called the *internal definition principle*. It says that if we start with a finite number of internal sets, and construct a new set from the others using operations which have first-order definitions, then the new set is also internal.

Before proving the Lemma, let's see an application:

**Example 2.1.** *If  $a, b \in {}^*\mathbb{R}$  and  $a^* < b$  then  $(a, b) := \{s \in {}^*\mathbb{R} \mid a^* < s^* < b\}$  is internal.*

Remark: An elementary application of transfer shows that  ${}^* <$ , the star-transform of the order relation on  $\mathbb{R}$ , is an actual linear order on  ${}^*\mathbb{R}$ . It follows that the interval  $(a, b)$  as defined above is the actual order-interval; this justifies the lack of a star on the parentheses. However, we will often drop such stars even without such justification, when writing them would be unduly tedious and not engender confusion.

Proof of Lemma 2.2:

(1) $\Rightarrow$ (2): Just like Proposition 2.1

(2) $\Rightarrow$ (1): Immediate from the definition

(1) $\Rightarrow$ (3): Take  $\phi$  to be  $x = x$  and  $B = E$ .

(3) $\Rightarrow$ (1): Let  $k$  be large enough that  $B, b_1, \dots, b_n \in {}^*V_k(S)$ . Define  $F : \mathbb{V}_k(S)^{n+1} \rightarrow \mathbb{V}_{k+1}(S)$  by  $F(c, c_1, \dots, c_n) := \{a \in c : \phi(a, c_1, \dots, c_n) \text{ is true in } \mathbb{V}(S)\}$ . By transfer one can verify (EXERCISE) that  $E = {}^*F(B, b_1, \dots, b_n)$ , which is an element of  ${}^*\mathbb{V}_{k+1}(S)$  by Lemma 2.1 and therefore internal.

### 3 Structure of the hyperreals

Since we assumed that  $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ , etc. are elements of  $\mathbb{V}(S)$ ,  ${}^*\mathbb{R}, {}^*\mathbb{Q}, {}^*\mathbb{Z}$ , and  ${}^*\mathbb{N}$  are in  $\mathbb{V}({}^*S)$ . What can we say about these sets?

Recall that a *group* is a triple  $\mathbf{G} = (G, \cdot, e)$  such that:

- (1)  $(\forall x \in G)(\exists y \in G)[(x \cdot y = e) \wedge (y \cdot x = e)]$
- (2)  $(\forall x \in G)[(x \cdot e = x) \wedge (e \cdot x = x)]$
- (3)  $(\forall x \in G)(\forall y \in G)(\forall z \in G)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$

(If  $\mathbf{G}$  only satisfies (1) and (2) above then it is a *monoid*.)

If  $\mathbf{G}$  is in  $\mathbb{V}(S)$  then the above are sentences in  $\mathcal{L}_{\forall}$ , and by transfer it follows:

- (1')  $(\forall x \in {}^*G)(\exists y \in {}^*G)[(x \cdot y = {}^*e) \wedge (y \cdot x = {}^*e)]$
- (2')  $(\forall x \in {}^*G)[(x \cdot {}^*e = x) \wedge ({}^*e \cdot x = x)]$
- (3')  $(\forall x \in {}^*G)(\forall y \in {}^*G)(\forall z \in {}^*G)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$

In other words,  ${}^*\mathbf{G}$  is also not only a  ${}^*$ group, but also an actual group, since the conditions for being a group and a  ${}^*$ group are *exactly the same!*

In fact, any internal triple  $(H, \times, \iota)$  satisfies the group axioms if and only if it satisfies the  ${}^*$ group axioms.

Similar results hold for pretty much all algebraic structures, e.g. Abelian groups, rings, fields, and ordered fields. The group  $(G, \cdot, e)$  is *Abelian* if it satisfies

$$(\forall x \in G)(\forall y \in G)[(x \cdot y) = (y \cdot x)]$$

A *ring* is a quadruple  $(F, +, \cdot, 0, 1)$  such that  $(F, +, 0)$  is an Abelian group,  $(F, \cdot, 1)$  is a monoid, and  $F$  satisfies the *distributive properties*

$$(\forall x \in F)(\forall y \in F)(\forall z \in F)[(x \cdot (y + z) = (x \cdot y) + (x \cdot z))]$$

$$(\forall x \in F)(\forall y \in F)(\forall z \in F)[((y + z) \cdot x) = (y \cdot x) + (z \cdot x)]$$

A *field* is a ring  $(F, +, \cdot, 0, 1)$  where  $(F \setminus \{0\}, \cdot, 1)$  is an Abelian group.

The field  $(F, +, \cdot, 0, 1)$  is *ordered* if it has a strict total order  $<$  such that

$$(\forall x \in F)(\forall y \in F)(\forall z \in F)[(x < y) \Rightarrow ((x + z) < (y + z))]$$

$$(\forall x \in F)(\forall y \in F)[((0 < x) \wedge (0 < y)) \Rightarrow (0 < x \cdot y)]$$

- Proposition 3.1.** 1. For every  $r \in {}^*\mathbb{R}$  there is an  $n \in {}^*\mathbb{Z}$  such that  $n \leq r < n + 1$
2. For every  $r \in {}^*\mathbb{R}$   $[r] \leq r < [r] + 1$
3. For every  $r \in {}^*(0, \infty)$  there is an  $n \in {}^*\mathbb{N}$  such that  $nr > 1$
4. etc.

Note that I have dropped the  $*$  from  $<$  and the greatest integer function  $[\ ]$ , and written  $nr$  for the horrible-but-more-correct  $n^* \cdot r$ .

*Proof.* Transfer □

**Proposition 3.2.** Let  $x \in {}^*\mathbb{R}$ . TFAE: (i)  $x \in \bigcap_{n \in \mathbb{N}}^* (-\frac{1}{n}, \frac{1}{n})$ ; (ii) For every (standard)  $\epsilon \in \mathbb{R}^+$ ,  ${}^*|x| < \epsilon$

**Definition 3.1.** Let  $x \in {}^*\mathbb{R}$ .

- a)  $x$  is infinitesimal if  ${}^*|x| < \epsilon$  for every  $\epsilon \in \mathbb{R}^+$
- b)  $x$  is finite if  ${}^*|x| < r$  for some  $r \in \mathbb{R}^+$
- c)  $x$  is infinite otherwise.

**Definition 3.2.** Let  $x, y \in {}^*\mathbb{R}$ .

Write  $x \approx y$  if  $x - y$  is infinitesimal.

Write  $x \approx 0$  if  $x$  is infinitesimal

Put  $\text{monad}(x) = \{y \in {}^*\mathbb{R} \mid x \approx y\}$

**Proposition 3.3.** Let  $\gamma, \delta \approx 0, x, y$  finite,  $\alpha, \beta$  infinite.

1.  $\gamma \pm \delta \approx 0, \gamma\delta \approx 0, \gamma x \approx 0, \alpha^{-1} \approx 0$
2. The following are finite:  $\gamma + x, x + y, 1/x$  (if  $x \not\approx 0$ )
3. The following are infinite:  $1/\gamma$  (if  $\gamma \neq 0$ ),  $|\alpha| + |\beta|, |\alpha| + x, |\alpha| + \gamma, |\alpha\beta|, |\alpha x|$  (if  $x \not\approx 0$ )

*Proof.* If  $r \in \mathbb{R}^+$  then  $-\frac{r}{2} < \gamma, \delta < \frac{r}{2}$ , so  $r < \gamma + \delta < r$ . As  $r$  is arbitrary,  $\gamma + \delta \approx 0$ .

For some  $M \in \mathbb{R}^+$ ,  $-M < x < M$ . If  $r \in \mathbb{R}$  then  $r + M < |\alpha|$ , so  $r - x < |\alpha|$ , consequently  $r < x + |\alpha|$ . Since  $r$  was arbitrary,  $x + |\alpha|$  is infinite.

The other assertions are proved in similar fashion. □

**Corollary 3.1.** If  $x \approx y$  and  $u \approx v$  then  $x \pm u \approx y \pm v$ . If  $x$  and  $u$  are finite then  $xu \approx yv$ .

The proof is left as an exercise.

From the above we see that the finite elements and infinitesimal elements of  ${}^*\mathbb{R}$  each form a subring, and that the infinitesimals are an ideal in the finite elements.

**Theorem 3.1.** (*Existence of the standard part*) *If  $x \in {}^*\mathbb{R}$  is finite then there is a unique  $r \in \mathbb{R}$  such that  $x \approx r$ .*

We call this  $r$  the *standard part* of  $x$ , and denote it by  ${}^{\circ}x$  or  $\text{st}(x)$ .

*Proof.* For uniqueness, note  $x = \text{st } x + \delta$  for some  $\delta = x - \text{st } x \approx x - x = 0$ . If also  $x = s + \gamma$  for  $s \in \mathbb{R}$  and  $\gamma \approx 0$  then  $s - \text{st } x = (x - \gamma) - (x - \delta) = \delta - \gamma \approx 0$ ; but two distinct real numbers cannot differ by an infinitesimal, so  $\text{st } x = s$ .

Now, put  $r = \sup\{a \in \mathbb{R} \mid a \leq x\}$ <sup>1</sup>, which exists because  $x$  is finite so the set is bounded in  $\mathbb{R}$ .

Let  $\epsilon$  be a standard positive real number; it suffices to show that  $x \in {}^*(r - \epsilon, r + \epsilon)$ . Since  $r - \epsilon < r$ , there is an  $a > r - \epsilon$  with  $a \leq x$ , so  $x > r - \epsilon$ . Suppose (for a contradiction) that  $x \geq r + \epsilon$ , then by definition of  $r$ ,  $r \geq r + \epsilon$ , a contradiction, so  $x < r + \epsilon$ , completing the proof.  $\square$

**Definition 3.3.**  $NS({}^*\mathbb{R}) = \{x \in {}^*\mathbb{R} \mid (\exists r \in \mathbb{R}) x \approx r\}$ , the nearstandard elements of  ${}^*\mathbb{R}$ . (*This is of course just the same thing as the finite elements of  ${}^*\mathbb{R}$ ; the notation will be useful later.*)

If  $r \in \mathbb{R}$ ,  $\text{monad}(r) = \{x \in {}^*\mathbb{R} \mid x \approx r\} = \text{st}^{-1}(r)$ .

We will often write  $NS(\mathbb{R})$  for  $NS({}^*\mathbb{R})$ , and more generally  $NS(E)$  for

**Proposition 3.4.** *Suppose  $x, y \in NS({}^*\mathbb{R})$ . (a)  ${}^{\circ}(x \pm y) = {}^{\circ}x \pm {}^{\circ}y$ ; (b)  ${}^{\circ}(xy) = {}^{\circ}x {}^{\circ}y$ ; (c) If  $y \neq 0$  then  ${}^{\circ}(\frac{x}{y}) = \frac{{}^{\circ}x}{{}^{\circ}y}$ ; and (d) If  $x \leq y$  then  ${}^{\circ}x \leq {}^{\circ}y$*

*Proof.* (c) Let  $\epsilon, \delta \approx 0$  with  $x = {}^{\circ}x + \epsilon, y = {}^{\circ}y + \delta$ . Then  $(\frac{{}^{\circ}x + \epsilon}{{}^{\circ}y + \delta} - \frac{{}^{\circ}x}{{}^{\circ}y}) = \frac{{}^{\circ}x {}^{\circ}y + \epsilon {}^{\circ}y - {}^{\circ}x {}^{\circ}y - \delta {}^{\circ}x}{{}^{\circ}y({}^{\circ}y + \delta)} = \underbrace{(\underbrace{\epsilon {}^{\circ}y}_{\approx 0} - \underbrace{\delta {}^{\circ}x}_{\approx 0})}_{\approx 0} \underbrace{\frac{1}{{}^{\circ}y({}^{\circ}y + \delta)}}_{\text{finite}} \approx 0$ .

The other parts are similar.  $\square$

The converse to part (d) of the above does not hold; if  $0 < \epsilon \approx 0$  then  $\epsilon < 2\epsilon$  but  ${}^{\circ}\epsilon = 0 = {}^{\circ}(2\epsilon)$ . However, it is the case that if  ${}^{\circ}x < {}^{\circ}y$  then  $x < y$ , in fact  $y - x > r$  for every standard  $r$  with  $0 < r < {}^{\circ}y - {}^{\circ}x$ .

**Corollary 3.2.** *TFAE: (i)  $\text{monad}(0) \neq \{0\}$ ; (ii) There are infinite hyperreal numbers; (c) There are nonstandard hyperreal numbers*

*Proof.* (i) $\Rightarrow$ (ii) If  $\epsilon \in \text{monad}0 \setminus \{0\}$  then (exercise)  $1/|\epsilon|$  is infinite.

(ii) $\Rightarrow$ (iii) Every infinite hyperreal is nonstandard

<sup>1</sup>Again, technically we should write  $r = \sup\{a \in \mathbb{R} \mid a^* \leq x\}$ , but will revert to the sloppier form for convenience.

(i) $\Rightarrow$ (ii) Exercise. □

Finally, a few observations about  ${}^*\mathbb{N}$ .

**Theorem 3.2.** 1. If  $H \in {}^*\mathbb{N} \setminus \mathbb{N}$  then  $H$  is infinite (an infinite integer)

2. If  $\mathbb{N}$  is internal (as a subset of  ${}^*\mathbb{N}$ ) then  $\mathbb{N} = {}^*\mathbb{N}$ . In particular, by Property 0 of the Fundamental Theorem,  $\mathbb{N}$  is external.

3. If  $E \subseteq {}^*\mathbb{N}$  is an internal set and (externally) infinite then  $E$  contains an infinite integer.

*Proof.* (1) Since  $(\forall x \in \mathbb{N})[x = 0 \wedge x > 0]$  holds in  $\mathbb{V}(S)$ ,  $(\forall x \in {}^*\mathbb{N})[x = 0 \wedge x > 0]$  holds in  $\mathbb{V}({}^*S)$ , so  $H \geq 0$ . Suppose (for a contradiction) that  $H$  is finite. Then the standard set of natural numbers  $\{n \in \mathbb{N} \mid H < n\}$  is nonempty, so has a least element  $m$ . Note  $m > 0$ . The sentence  $(\forall x \in \mathbb{N})[\neg(m - 1 < x < m)]$  holds in  $\mathbb{V}(S)$ , so  $(\forall x \in {}^*\mathbb{N})[\neg(m - 1 < x < m)]$  holds in  $\mathbb{V}({}^*S)$ . Since  $H \in {}^*\mathbb{N}$  and  $x < m$ ,  $x \leq m - 1$ . Since  $H \notin \mathbb{N}$ ,  $H \neq m$ , so  $H < m - 1$ , contradicting the choice of  $m$ .

(2) Recall the principle of mathematical induction:

$$(\forall x \in \mathcal{P}(\mathbb{N}))[(0 \in x) \wedge (\forall n \in \mathbb{N})(n \in x \Rightarrow n + 1 \in x) \Rightarrow (\forall y \in \mathbb{N})(y \in x)]$$

Therefore,

$$(\forall x \in {}^*\mathcal{P}(\mathbb{N}))[(0 \in x) \wedge (\forall n \in {}^*\mathbb{N})(n \in x \Rightarrow n + 1 \in x) \Rightarrow (\forall y \in {}^*\mathbb{N})(y \in x)]$$

Since the hypotheses (on  $x$ ) of the last statement are satisfied by  $\mathbb{N}$ ,  ${}^*\mathbb{N} \subseteq \mathbb{N}$ .

(3) If  $E \not\subseteq \mathbb{N}$  then by (1) any element of  $E \setminus \mathbb{N}$  is infinite. Otherwise, put  $N = \{x \in {}^*\mathbb{N} \mid (\exists e \in E)[x \leq e]\}$ .  $N$  is internal by the internal definition principle. Since  $E$  is infinite,  $\mathbb{N} \subseteq E \subseteq \mathbb{N}$ , so  $\mathbb{N}$  is internal, contradicting (2). □

## 4 Some saturation-like properties

**Theorem 4.1.** *Let  $A$  be an internal subset of  ${}^*\mathbb{R}$ .*

1. **Overflow:** *If  $A$  contains arbitrarily large finite numbers, then it contains an infinite number.*
2. **Overspill:** *If  $A$  contains every infinitesimal, then for some standard  $r \in \mathbb{R}$ , the interval  $(-r, r) \subseteq A$*
3. **Underflow:** *If  $A$  contains arbitrarily small infinite numbers, then it contains a finite number.*
4. **Underspill:** *If  $A$  contains arbitrarily small noninfinitesimals, then it contains an infinitesimal.*

*Proof.* **I. Proofs without using saturation**

1. Let  $N = \{n \in {}^*\mathbb{N} : (\exists a \in A)[n < a]\}$ . Then  $N$  is internal, and by hypothesis contains all standard elements of  $\mathbb{N}$ , so contains an infinite integer  $n$ . Any  $a \in A$  with  $n < a$  will be infinite.
2. The set  $\{s \in {}^*\mathbb{R} : (-|s|, |s|) \subseteq A\}$  is internal (why?), and by hypothesis contains all infinitesimals, so (since  $\text{monad}(0)$  is not internal) contains a noninfinitesimal  $s$ . Now take  $r$  standard with  $0 < r < \text{st } |s|$ .  
Note that overflow also follows from overspill: WOLG  $0 \notin A$ . If overflow fails then  $1/a \not\approx 0$  for every  $a \in A$ . The set  $\{x \in {}^*\mathbb{R} : (\forall a \in A)[|x| < |1/a|]\}$  contains all infinitesimals, and therefore an interval  $(-r, r)$  for some standard  $r$ . It follows that  $A$  is bounded by  $2/r$ , a contradiction.
3. WOLG  $0 \notin A$ . Let  $B = \bigcup_{a \in A} (-1/|a|, 1/|a|)$ .  $B$  is internal (why?), and if  $\epsilon \approx 0$  then by hypothesis  $|a| < 1/|\epsilon|$  for some  $a \in A$ ; it follows that  $|\epsilon| < 1/|a|$ , so is in  $B$ . By overspill  $B$  contains the star of a standard interval, hence some standard  $r > 0$ . If  $a$  is chosen so that  $r < 1/|a|$  then  $|a| < 1/r$  so is a finite number.
4. WOLG  $A \subseteq {}^*[-1, 1]$  and  $0 \notin A$ . Put  $B = \{1/a : a \in A\}$ ,  $B$  is an internal set containing arbitrarily large finite number, so it contains an infinite number  $1/a$ , so  $a \approx 0$ .

□

*Proof.* **II. Proofs using saturation**

1. Let  $A_n = A \cap ((-\infty, n) \cup (n, \infty))$  for  $n \in \mathbb{N}$ . By hypothesis each  $A_n$  is nonempty, and  $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ , so by  $\aleph_1$ -saturation  $\exists a \in \bigcap_{n \in \mathbb{N}} A_n$ . This  $a$  is evidently infinite.
2. Otherwise for every  $0 \neq n \in \mathbb{N}$ , the set  $A_n = {}^*(-1/n, 1/n) \setminus A$  would be nonempty. Since  $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ , by  $\aleph_1$ -saturation  $\exists \epsilon \in \bigcap_{n \in \mathbb{N}^+} A_n$ , which is evidently an infinitesimal not in  $A$ , a contradiction.

3. Exercise.

4. As in (2), but put  $A_n = A \cap^* (-1/n, 1/n)$

□

For the next properties,  $\aleph_1$ -saturation might not be sufficient. Therefore:

ASSUMPTION:  $\kappa > \text{card}(\mathbb{V}(S))$

Recall that  $\kappa$  was a cardinal given in the Fundamental Theorem. This section of that theorem is often stated not in terms of  $\kappa$  but in terms of families of internal sets indexed (externally) by standard sets; the two formulations are equivalent, but the latter can be a bit confusing, since it suggests that the standard set itself (not just its cardinality) is somehow important.

Often much less saturation suffices. The following properties, which we will prove using saturation, were true for nonstandard models constructed before the role of saturation was clarified by W.A.J. Luxemburg, and used by early authors in place of saturation arguments.

**Definition 4.1.** *A (standard) relation  $R$  is concurrent provided for every  $n \in \mathbb{N}$  and  $a_1, \dots, a_n$  in the domain of  $R$  there is a  $b$  such that  $a_i R b$  for all  $i \leq n$ .*

For example,  $<$  on  $\mathbb{N}$  and  $\mathbb{R}$ , and  $\subseteq$  on  $\mathcal{P}_f(R) = \{A \subseteq \mathbb{R} : A \text{ is finite}\}$ , are concurrent relations.

**Theorem 4.2.** *Let  $R \in \mathbb{V}(S)$  be a standard concurrent relation. Then there is a  $b$  in the range of  ${}^*R$  such that for every  $a$  in the domain of  $R$ ,  $a {}^*R b$ .*

*Proof.* By concurrence, the statements

$$“a {}^*R x” \quad (a \in \text{dom}(R))$$

are *finitely satisfiable* (that is, for every finite set of them there is an element  $b$  of the range of  ${}^*R$  making the conjunction of the statements true in  $\mathbb{V}({}^*S)$  when  $b$  is substituted for  $x$ ), and so by  $\kappa$ -saturation,  $\kappa > \text{card } A$ , there is a single element  $b$  satisfying all such conditions. □

The saturation argument just given is easily translated into an argument about a family of internal sets with the finite intersection property; the reader is encouraged to do the translation. It will frequently be more natural to formulate arguments in this manner.

The next property prompted Michael Richter to describe Nonstandard Analysis as “the art of making infinite sets finite by extending them.”

**Theorem 4.3.** *Let  $A$  be any set in  $\mathbb{V}(S)$ . There is a set  $\hat{A}$  in  $\mathbb{V}({}^*S)$  such that (i)  $\hat{A} \subseteq {}^*A$ ; (ii)  $A \subseteq \hat{A}$  (in the sense that for every  $a \in A$ ,  ${}^*a \in \hat{A}$ ); and (iii)  $\hat{A}$  is *{}^\*-finite* (or hyperfinite).*

A set is *hyperfinite* if it is internally finite; that is, it is the image of an internal function with domain  $\{0, 1, 2, \dots, H\}$  for some  $H \in {}^*\mathbb{N}$ ; equivalently, if it is in  ${}^*\mathcal{P}_f(X)$  for some internal  $X$ , where  $\mathcal{P}_f(E)$  is the set of finite subsets of  $E$ .

*Proof.* The following conditions on  $X$ :

$$X \in {}^*\mathcal{P}_f(A)$$

$${}^*a \in X, \quad (a \in A)$$

are evidently finitely satisfiable, so by saturation there is some  $\hat{A}$  that satisfies all of them. (Alternately, use concurrence.)  $\square$

**Theorem 4.4.** (*Prolongation*)

1. Let  $\{s_n\}_{n \in {}^*\mathbb{N}}$  be an internal sequence in  ${}^*\mathbb{R}$ , and suppose  $|s_n| < M$  for all finite  $n$ . Then there is an infinite  $H \in {}^*\mathbb{N}$  such that  $|s_n| < M$  for all  $n \leq H$ .
2. Let  $s_n \in {}^*\mathbb{R}$  for every  $n \in \mathbb{N}$ ; then there is an internal sequence  $\{t_n\}_{n \in {}^*\mathbb{N}}$  such that  $s_n = t_n$  for every standard  $n$ .
3. Let  $\{s_n\}_{n \in {}^*\mathbb{N}}$  be an internal sequence in  $\mathbb{R}$ , and suppose  $|s_n| \approx 0$  for all finite  $n$ . Then there is an infinite  $H \in {}^*\mathbb{N}$  such that  $|s_n| \approx 0$  for all  $n \leq H$ .

*Proof.* 1. Apply overflow to  $\{m \in {}^*\mathbb{N} : (\forall n \leq m)[|s_n| \leq M]\}$

2. Exercise

3. First, note that we cannot simply apply the same argument as in (1) where  $\leq M$  is replaced by  $\approx 0$ , since then the set in question would not be internal. Instead apply overflow to the internal set  $\{m \in {}^*\mathbb{N} : (\forall n \leq m)[|s_n| \leq 1/m]\}$ .

$\square$

## 5 Calculus

**Theorem 5.1.** Let  $\{s_n\}_{n \in \mathbb{N}}$  be a sequence of real numbers

1.  $\lim_{n \rightarrow \infty} s_n = a$  if and only if for every infinite  $k \in {}^*\mathbb{N}$ ,  $s_k \approx a$
2.  $\lim_{n \rightarrow \infty} s_n = \infty$  if and only if for every infinite  $k \in {}^*\mathbb{N}$ ,  $s_k$  is positive infinite
3.  $\{s_n\}_{n \in \mathbb{N}}$  is a bounded sequence if and only if for every infinite  $k \in {}^*\mathbb{N}$ ,  $s_k$  is finite
4.  $\{s_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence if and only if for every infinite  $m, n \in {}^*\mathbb{N}$ ,  $s_m \approx s_n$
5.  $r$  is a limit point of  $\{s_n\}_{n \in \mathbb{N}}$  if and only if there exists an infinite  $k \in {}^*\mathbb{N}$  with  $s_k \approx r$

*Proof.* 1.  $\Rightarrow$ ) Let  $k$  be infinite. Let  $\epsilon > 0$  be standard, then there is an  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $|s_n - a| < \epsilon$ . It follows by transfer that  $|s_k - a| < \epsilon$ . Since  $\epsilon$  was arbitrary,  $s_k \approx a$ .

$\Leftarrow$ )  $\epsilon > 0$  standard. The statement  $(\exists N \in {}^*\mathbb{N})(\forall k > N)[|s_k - a| < \epsilon]$  is true in  $\mathbb{V}({}^*\mathbb{S})$  (in fact, *any* infinite  $N$  works), and the result follows by transfer.

2. Similar to the proof of (1)
3. If  $\{s_n\}_{n \in \mathbb{N}}$  is bounded by  $M$  then  $(\forall n \in \mathbb{N})[|s_n| < M]$  holds, and by transfer  $|s_n| < M$  for any infinite  $k$  as well. Conversely, if  $\{s_n\}_{n \in \mathbb{N}}$  is *not* bounded then  $(\forall M \in \mathbb{N})(\exists n \in \mathbb{N})[|s_n| > M]$ . By transfer, if  $M \in {}^*\mathbb{N}$  is infinite then for some  $n \in {}^*\mathbb{N}$ ,  $|s_n| > M$  so is infinite.
4. Exercise
5. Exercise

□

**Corollary 5.1.** If  $\{s_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence then  $\lim_{n \rightarrow \infty} s_n$  exists

*Proof.* Fix an infinite  $k \in {}^*\mathbb{N}$ . There is a standard  $N$  such that for all  $m, n \geq N$ ,  $|s_m - s_n| < 1$ . By transfer,  $|s_k - s_N| < 1$ , so  $s_k$  is bounded and has a standard part  ${}^{\circ}s_k$ . Let  $n$  be infinite. By the previous theorem,  $s_n \approx s_k$ . It follows that  $s_n \approx {}^{\circ}s_k$ . Since  $n$  was arbitrary in  ${}^*\mathbb{N} \setminus \mathbb{N}$ , by the previous theorem the sequence has a limit of  ${}^{\circ}s_k$ . □

**Corollary 5.2.** If  $\lim_{n \rightarrow \infty} a_n = a$ ,  $\lim_{n \rightarrow \infty} b_n = b$  then (i)  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ ; (ii)  $\lim_{n \rightarrow \infty} (a_n - b_n) = a - b$ ; (iii)  $\lim_{n \rightarrow \infty} (a_n b_n) = ab$ ; if  $b \neq 0$  then  $\lim_{n \rightarrow \infty} (a_n/b_n) = a/b$

*Proof.* (iii) If  $n$  is infinite then  ${}^{\circ}(a_n/b_n) = ({}^{\circ}a_n/{}^{\circ}b_n) = (a/b)$ . □

**Corollary 5.3.** Let  $\{a_n\}_{n \in \mathbb{N}}$  be a (standard) sequence in  $\mathbb{R}$ , and let  $s \in \mathbb{R}$ .

1.  $\sum_n a_n = s$  if and only if  ${}^*\sum_{n=0}^N a_n \approx s$  for every infinite  $N \in {}^*\mathbb{N}$
2.  $\sum_n a_n$  converges if and only if  ${}^*\sum_{n=N}^M a_n \approx 0$  for every infinite  $N \leq M \in {}^*\mathbb{N}$
3. If  $\sum_n a_n$  converges then  $\lim_{n \rightarrow \infty} a_n = 0$

*Proof.* (1) is just Theorem 5.1. (2) follows immediately from (1) and  ${}^*\sum_{n=N}^M a_n = {}^*\sum_{n=0}^M a_n - {}^*\sum_{n=0}^{N-1} a_n$ . (3) is (2) with  $M = N$ .  $\square$

**Theorem 5.2.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$ , and  $c \in I$ .

1.  $\lim_{x \rightarrow c} f(x) = L$  provided  $(\forall x \in {}^*I)[c \neq x \approx c \Rightarrow {}^*f(x) \approx L]$
2.  $f$  is continuous at  $c$  provided  $(\forall x \in {}^*I)[x \approx c \Rightarrow {}^*f(x) \approx f(c)]$ . (In other words,  ${}^*f(\text{monad}(c)) \subseteq \text{monad}(f(c))$ .)

*Proof.* (2) Suppose that  $f$  is continuous at  $c$ , and  $z \approx c$ . Fix  $\epsilon > 0$  standard. There is a standard  $\delta > 0$  such that whenever  $0 \neq |z - c| < \delta$ ,  $|f(z) - f(c)| < \epsilon$ . By transfer and the fact that  $|x - c| < \delta$ ,  $|{}^*f(x) - f(c)| < \epsilon$ . Since  $\epsilon > 0$  is arbitrary,  ${}^*f(x) \approx f(c)$ .

Conversely, suppose the putative nonstandard condition for continuity holds, and that  $\epsilon > 0$  is standard. Any positive infinitesimal  $\delta$  satisfies  $(\exists \delta \in {}^*\mathbb{R}^+)(\forall x \in {}^*I)[|x - c| < \delta \Rightarrow |{}^*f(x) - f(c)| < \epsilon]$ . The result now follows by transfer.  $\square$

Continuity on the interval  $I$  becomes:  $(\forall x \in I)(\forall y \in {}^*I)[x \approx y \Rightarrow f(x) \approx {}^*f(y)]$ . What happens if we replace the  $I$  by its star?

**Theorem 5.3.** Let  $I \subseteq \mathbb{R}$  be an interval and  $f : I \rightarrow \mathbb{R}$ . Then  $f$  is uniformly continuous on  $I$  if and only if  $(\forall x, y \in {}^*I)[x \approx y \Rightarrow {}^*f(x) \approx {}^*f(y)]$

*Proof.* ( $\Leftarrow$ ) let  $\epsilon > 0$ , and consider

$$E = \{\delta \in {}^*\mathbb{R}^+ | (\forall x, y \in {}^*I)[|x - y| < \delta \Rightarrow |{}^*f(x) - {}^*f(y)| < \epsilon]\}$$

this set is internal, and contains all positive infinitesimals, so must contain at least one positive noninfinitesimal, so some positive real number  $\delta$ . This  $\delta$  witnesses uniform continuity.

( $\Rightarrow$ ) Let  $x \approx y \in {}^*I$ . Let  $\epsilon > 0$  be standard. By uniform continuity the set  $E$  above contains at least one standard  $\delta$ ; since  $|x - y| < \delta$ ,  $|{}^*f(x) - {}^*f(y)| < \epsilon$ . Since  $\epsilon$  was arbitrary,  ${}^*f(x) \approx {}^*f(y)$ .  $\square$

**Corollary 5.4.** If  $f$  is continuous on the interval  $[a, b]$  then  $f$  is uniformly continuous on  $[a, b]$

*Proof.* First, note that that by proposition 3.4, for any  $x \in {}^*[a, b]$ ,  ${}^{\circ}x \in [a, b]$ . If  $x \approx y \in {}^*[a, b]$  then  ${}^*f(x) \approx f({}^{\circ}x) = f({}^{\circ}y) \approx {}^*f(y)$ .  $\square$

As an example, consider the function  $f(x) = \frac{1}{x}$  on  $(0, 1)$ . If  $H \in {}^*\mathbb{N} \setminus \mathbb{N}$  then  $\frac{1}{H} \approx 0 \approx \frac{1}{H^2}$  but  ${}^*f(\frac{1}{H}) - {}^*f(\frac{1}{H^2}) = H^2 - H$ , which is infinite, let alone infinitesimal.

Before continuing, it will be useful to briefly discuss *hyperfinite sets*. For  $X$  and set, write  $\mathcal{P}_f(X) = \{A \subseteq X \mid A \text{ is finite}\}$ .

**Proposition 5.1.** *Suppose  $E \in \mathbb{V}({}^*S)$ . TFAE: (1)  $(\exists n \in \mathbb{N})(E \in {}^*\mathcal{P}_f(\mathbb{V}_n(S)))$ ; (2) For some internal  $X$ ,  $E \in {}^*\mathcal{P}_f(X)$ ; (3) For some  $H \in {}^*\mathbb{N}$  there is an internal bijection from  $\{0, 1, 2, \dots, H\}$  onto  $E$*

We call such a set  $E$  a *hyperfinite set*.

*Proof.* (1  $\Rightarrow$  2) Let  $X = {}^*\mathbb{V}_n(S)$ . (2  $\Rightarrow$  3) Transfer. (3  $\Rightarrow$  1) Note  $E$  must be internal, so for some standard  $n \in \mathbb{N}$  the sets  $E, f$ , and  $\{0, 1, 2, \dots, H\}$  are all elements of  ${}^*\mathbb{V}_n(S)$ . Now apply transfer.  $\square$

**Theorem 5.4.** (*Extreme Value Theorem*) *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function; then  $f$  attains both its maximum and minimum values on  $[a, b]$*

*Proof.* Let  $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ , and let  $\Delta x = (b - a)/H$ . Let  $\Omega = \{a, a + \Delta x, a + 2\Delta x, \dots, b\} = \{x \in {}^*[a, b] \mid (\exists k \in {}^*\mathbb{N})[x = a + k(b - a)/H]\}$ . Note  $\Omega \in {}^*\mathcal{P}_f([a, b])$ . By transfer of the statement “if  $f$  is a real-valued function on a finite subset of  $[a, b]$  then  $f$  attains a maximum on the set,” there must be  $c \in \Omega$  such that  ${}^*f$  is maximized at  $c$  on  $\Omega$ . It remains to show that  $f$  has a maximum at  ${}^{\circ}c$  on  $[a, b]$ .

If  $x \in [a, b]$  then for some  $\omega \in \Omega, \omega \leq x \leq \omega + \Delta x$ . In particular,  $x \approx \omega$ . By continuity,  $f(x) \approx {}^*f(\omega) \leq {}^*f(c)$  (by choice of  $c$ )  $\approx f({}^{\circ}c)$ . Since  $f(x) \lesssim f({}^{\circ}c)$ , and these are both standard real numbers,  $f(x) \leq f({}^{\circ}c)$ . Since  $x$  was arbitrary,  $f$  has a maximum at  ${}^{\circ}c$  on  $I$ . A similar argument shows that  $f$  has a minimum on  $I$ .  $\square$

Note that in this proof it was easy to specify the point where  $f$  was maximized, more difficult to verify that  $f$  really did have a maximum there. This is a common feature of nonstandard existence proofs: finding the object is often straightforward, the work is in showing that this “obvious” object is the one desired.

**Theorem 5.5.** (*Intermediate Value Theorem*) *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function, with  $f(a) < d < f(b)$  then  $f(c) = d$  for some  $c \in (a, b)$*

*Proof.* Let  $H, \Delta x$ , and  $\Omega$  as in the proof of Theorem 5.4. By transfer of the comparable property for finite subsets of  $[a, b]$ , there is an  $\omega \in \Omega$  such that  ${}^*f(\omega) \leq d \leq {}^*f(\omega + \Delta x)$ . (For example, let  $\omega$  be the  ${}^*$ -greatest element of  $\Omega$  such that  $(\forall \gamma \in \Omega)[\gamma \leq \omega \Rightarrow {}^*f(\gamma) \leq d]$ ). Let  $c = {}^{\circ}\omega = {}^{\circ}(\omega + \Delta x)$ . By continuity,  $f(c) \lesssim d \lesssim f(c)$ , and since these are all standard numbers,  $f(c) = d$ .  $\square$

We now prove the following remarkable theorem of Darboux:

**Theorem 5.6.** *Suppose  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  satisfies the additive identity*

$$(\forall x, y \in \mathbb{R})[\phi(x + y) = \phi(x) + \phi(y)]$$

*Suppose in addition that  $\phi$  is bounded on some interval. Then*

$$(\forall x \in \mathbb{R})[\phi(x) = x\phi(1)].$$

*In particular,  $\phi$  is linear.*

*Proof.* From

$$\phi(0) = \phi(0 + 0) = \phi(0) + \phi(0)$$

we see that  $\phi(0) = 0$ . For any  $x$ ,  $\phi(x) + \phi(-x) = \phi(x + -x) = \phi(0) = 0$ , so  $\phi(-x) = -\phi(x)$ .

If  $m, n \in \mathbb{Z}^+$  and  $c \in \mathbb{R}$  then

$$m\phi(c) = \phi(\underbrace{c + c + c + \cdots + c}_m) = \phi(mc) = \phi(\underbrace{\frac{m}{n}c + \frac{m}{n}c + \cdots + \frac{m}{n}c}_n) = n\phi(\frac{m}{n}c)$$

so  $\phi(\frac{m}{n}c) = \frac{m}{n}\phi(c)$ . Combined with the results above,  $\phi(xc) = x\phi(c)$  for all rational  $x$  and arbitrary  $c$ . (By transfer, we also have that  $(\forall x \in {}^*\mathbb{Q})(\forall c \in {}^*\mathbb{R})[{}^*\phi(xc) = x{}^*\phi(c)]$ ) In particular,  $\phi(x) = x\phi(1)$  for all rational  $x$ .

To extend this result to all *real*  $x$  it suffices to prove that  $\phi$  is continuous on  $\mathbb{R}$ . We'll prove *uniform* continuity. Let  $x \approx y$  in  ${}^*\mathbb{R}$ . WOLG  $x < y$ . We're given that  $\phi$  is bounded on some interval  $(a, b)$ . We may suppose that  $a > 0$  and  $|\phi| < M$  on  $(a, b)$ .

Let  $H \in {}^*\mathbb{N}$  be least with  $H(y - x) > a$ ; such an  $H$  exists by transfer of the Archimedean principle. Since  $x \approx y$  and  $a > 0$  is standard,  $H$  must be infinite, and  $H(y - x) < b$ . Then  $|H({}^*\phi(y) - {}^*\phi(x))| = |H({}^*\phi(y - x))| = |{}^*\phi(H(y - x))| < M$ , so  $|{}^*\phi(y) - {}^*\phi(x)| < M/H \approx 0$ , proving continuity.  $\square$

There are other conditions, in place of the condition that  $\phi$  is bounded on some interval, which also guarantee that an additive function is linear. Such conditions include  $\phi$  being continuous at some point, or measurable. You should be able to verify that these also imply linearity, also that in the absence of such a condition  $\phi$  need *not* be linear.

## 5.1 Differentiation

**Theorem 5.7.** Let  $f : (a, b) \rightarrow \mathbb{R}, c \in (a, b), D \in \mathbb{R}$ . TFAE:

1.  $f'(c) = D$
2. For any  $\Delta x \approx 0, \Delta x \neq 0, \frac{*f(x+\Delta x) - *f(x)}{\Delta x} \approx D$
3. For any  $\Delta x \approx 0$  there is an  $\epsilon \approx 0$  such that  $*f(x + \Delta x) = *f(x) + D\Delta x + \epsilon\Delta x$

*Proof.* (1) $\Leftrightarrow$ (2) is immediate by the nonstandard characterization of limits. (2) $\Rightarrow$ (3) If  $\Delta x = 0$  let  $\epsilon = 0$ . Otherwise, let  $\epsilon = \frac{*f(x+\Delta x) - *f(x)}{\Delta x} - D$ . (3) $\Rightarrow$ (2) Divide both sides of (3) by  $\Delta x$ .  $\square$

**Corollary 5.5.** Let  $f : (a, b) \rightarrow \mathbb{R}, c \in (a, b)$ . If  $f'(c)$  exists then  $f$  is continuous at  $c$ .

*Proof.* If  $x \approx c$  put  $\Delta x = x - c \approx 0$ , then  $*f(x) = *f(c + \Delta x) = *f(c) + \underbrace{f'(c)\Delta x}_{\approx 0} + \underbrace{\epsilon\Delta x}_{\approx 0} \approx f(c)$   $\square$

**Example 5.1.** Let  $f, g : (a, b) \rightarrow \mathbb{R}$  be differentiable at  $c \in (a, b)$ ; then so is  $fg$ , and  $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

*Proof.*  $*f(c + \Delta x)*g(c + \Delta x) = (f(c) + f'(c)\Delta x + \epsilon\Delta x)(g(c) + g'(c)\Delta x + \delta\Delta x) = f(c)g(c) + (f'(c)g(c) + f(c)g'(c))\Delta x + \underbrace{(\epsilon g(c) + \epsilon g'(c)\Delta x + \epsilon\delta\Delta x + \delta f(c) + \delta f'(c)\Delta x)}_{\approx 0} \Delta x$   $\square$

**Theorem 5.8.** (Mean Value Theorem) Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous,  $f$  differentiable on  $(a, b)$ . For some  $c \in (a, b), f'(c) = \frac{f(b) - f(a)}{b - a}$

*Proof.* As usual, it suffices to prove Rolle's Theorem: If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous,  $f$  differentiable on  $(a, b)$ , and  $f(a) = f(b) = 0$ . Then for some  $c \in (a, b), f'(c) = 0$ . (To get the general case, apply Rolle to  $g(x) = f(x) - L(x)$ , where  $y = L(x)$  is the equation of the line connecting  $(a, f(a))$  and  $(b, f(b))$ .) To prove Rolle's Theorem, consider three cases:

$f \equiv 0$  on  $[a, b]$  Then any  $c \in (a, b)$  works.

$f(x) > 0$  for some  $x \in [a, b]$  Let  $c \in [a, b]$  be a point at which  $f$  has a maximum on  $[a, b]$ , note that  $c \neq a, b$ . Let  $\delta \approx 0, \delta > 0$ . Then  $0 \leq \frac{*f(c-\delta) - *f(c)}{-\delta} \approx f'(c) \approx \frac{*f(c+\delta) - *f(c)}{\delta} \leq 0$ . The result follows by recalling that the standard part approximately preserves order.

$f(x) < 0$  for some  $x \in [a, b]$  Apply the last case to  $-f$ .  $\square$

**Definition 5.1.** Let  $I$  be an interval in  $\mathbb{R}$ ,  $c \in I$ .  $f : I \rightarrow \mathbb{R}$  has a local maximum at  $c \in I$  provided for all  $x \approx c$ ,  $x \in I \Rightarrow^* f(x) \leq f(c)$ .  $f$  has a local minimum at  $c \in I$  provided for all  $x \approx c$ ,  $x \in I \Rightarrow^* f(x) \geq f(c)$ .

**Exercise 5.1.** Show that this is equivalent to the usual definition.

**Corollary 5.6.** If  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable at  $c \in (a, b)$ , and  $f$  has a local maximum or minimum at  $c$ , then  $f'(c) = 0$

*Proof.* This follows from the above proof of Rolle's Theorem.  $\square$

**Theorem 5.9.** (Chain Rule) Let  $g : (a, b) \rightarrow \mathbb{R}$  be differentiable at  $c \in (a, b)$ ,  $I$  be an interval containing the range of  $g$ , and  $f : I \rightarrow \mathbb{R}$  differentiable at  $g(c)$ . Then  $f \circ g$  is differentiable at  $c$ , and  $(f \circ g)'(c) = f'(g(c))g'(c)$ .

*Proof.* Let  $\Delta x \approx 0$ , then

$$(f \circ g)(c + \Delta x) = f(g(c + \Delta x)) = f(g(c) + \underbrace{g'(c)\Delta x + \epsilon\Delta x}_{:=\Delta y \approx 0})$$

(for some  $\epsilon \approx 0$ )

$$= f(g(c)) + f'(g(c))\Delta y + \delta\Delta y = f(g(c)) + f'(g(c))g'(c)\Delta x + \underbrace{\delta(g'(c) + \epsilon)}_{\approx 0}\Delta x$$

(for some  $\delta \approx 0$ ).  $\square$

## 5.2 Uniform differentiability

**Lemma 5.1.** Let  $I$  be an interval,  $f, \phi : I \rightarrow \mathbb{R}$ . TFAE:

$$1. (\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in I)[0 \neq |x - y| < \delta \Rightarrow \left| \frac{f(y) - f(x)}{y - x} - \phi(x) \right| < \epsilon]$$

$$2. (\forall x \neq y \in {}^*I)[x \approx y \Rightarrow \frac{{}^*f(y) - {}^*f(x)}{y - x} \approx {}^*\phi(x)]$$

*Proof.* Exercise  $\square$

**Definition 5.2.**  $f : I \rightarrow \mathbb{R}$  is uniformly differentiable (UD) on  $I = (a, b)$  provided for some  $\phi$  the equivalent conditions above hold.

**Proposition 5.2.** If  $f$  is UD on  $I$  then  $f$  is differentiable on  $I$  with  $f' = \phi$

*Proof.* Exercise  $\square$

**Theorem 5.10.** Let  $f : I \rightarrow \mathbb{R}$ . TFAE:

1.  $f$  is UD on  $I$ .

2.  $f'$  exists and is uniformly continuous on  $I$

*Proof.* (1  $\Rightarrow$  2) Let  $\phi$  witness UD. If  $x \approx y \in {}^*I$  then  $\phi(x) \approx \frac{{}^*f(y) - {}^*f(x)}{y - x} = \frac{{}^*f(x) - {}^*f(y)}{x - y} \approx \phi(y)$ .

(2  $\Rightarrow$  1) Let  $x \approx y \in {}^*I$ ,  $x < y$ . By the \*Mean value Theorem there is a  $c \in (x, y)$  with  $\frac{{}^*f(y) - {}^*f(x)}{y - x} = {}^*f'(c)$ , which  $\approx {}^*f'(x)$  and  $\approx {}^*f'(y)$  by uniform continuity of  $f'$ , since  $x \approx c \approx y$ .  $\square$