General Topology

February 5, 2018

1 Basic Concepts

A topology on a set X is a collection $\mathfrak{T}$ of sets, called *open sets*, such that $\emptyset, X \in \mathfrak{T}$ and $\mathfrak{T}$ is closed under finite intersection and arbitrary union. We call the pair $(X, \mathfrak{T})$ a *topological space*. For example, the *standard topology* on $\mathbb{R}$ consists of all sets obtained as a union of a set of open intervals.

The complement of an open set is a *closed set*.

Many topological concepts have simple and natural nonstandard characterizations; we had a taste of this in the earlier chapters. As in $\mathbb{R}$, the fundamental tool in nonstandard topology is the *monad*.

Definition 1.1. *Let $(X, \mathfrak{T})$ be a topological space, and $p \in X$. The* monad *of p is the set $m(p) = \bigcap\{{}^*u : p \in u \in \mathfrak{T}\}$.*

Note that we only define monads of *standard* elements. An element q of ${}^*X$ is called *nearstandard* if $q \in m(p)$ for some $p \in X$, and the set of all nearstandard elements of ${}^*X$ is denoted by $NS({}^*X)$. More generally, we will write $NS({}^*E)$ for $\bigcup_{e \in E} m(e)$ when $E \subseteq X$. When we are discussing more than one topological space, or two topologies on the same space, we will sometimes write $m_X(x)$ or $m_{\mathfrak{T}}(x)$ to eliminate ambiguity.

Lemma 1.1. *For any $p \in X$ there is a $u \in {}^*\mathfrak{T}$ with $p \in u \subseteq m(p)$.*

Such a u is called an *infinitesimal neighborhood* of p.

Proof. The conditions "$p \in u \in {}^*\mathfrak{T}$" and "$u \subseteq {}^*v$" ($v \in \mathfrak{T}$) on u are evidently finitely satisfiable, so there is a $u \in {}^*\mathfrak{T}$ satisfying all of them. Alternately, if $\hat{\mathfrak{T}}$ is a hyperfinite subset of ${}^*\mathfrak{T}$ with $\mathfrak{T} \subseteq \hat{\mathfrak{T}} \subseteq {}^*\mathfrak{T}$ then $u = \bigcap\{u \in \hat{\mathfrak{T}} : p \in u\} \in {}^*\mathfrak{T}$ (since $\mathfrak{T}$ is closed under finite intersections). ⊣

A *basis* for a topology $\mathfrak{T}$ is a set $\mathcal{B} \subset \mathfrak{T}$ such that for every $x \in u \in \mathfrak{T}$ there is a $v \in \mathcal{B}$ with $x \in v \subseteq u$. the elements of $\mathcal{B}$ are called *basic open sets*. $\mathcal{B}$ *generates* $\mathfrak{T}$ in the sense that a set is open if and only if it is the union of the basic open sets contained in it.

Lemma 1.2. *Let $\mathcal{B}$ be a* basis *for a topology $\mathfrak{T}$. For any $p \in X, m(p) = \bigcap\limits_{p \in u \in \mathcal{B}} {}^*u$*

The proof is left to the reader.

Suppose $\mathcal{B}$ is a subset of $\mathcal{P}(X)$ such that (i) $X \subseteq \bigcup \mathcal{B}$ and (ii) for any $u, v \in \mathcal{B}$ and $x \in u \cap v$ there is a $w \in \mathcal{B}$ with $x \in w \subseteq u \cap v$. It is easy to verify that there is a unique topology $\mathfrak{T}$ for which $\mathcal{B}$ is a basis.

Lemma 1.3. *Let $(X, \mathfrak{T})$ be a topological space. The following are equivalent:*

1. *$(X, \mathfrak{T})$ is Hausdorff (or T_2); that is, whenever $x, y \in X$, if $x \neq y$ then there are disjoint open sets u and v with $x \in u$ and $y \in v$.*
2. *For every $x, y \in X$, if $x \neq y$ then $m(x) \cap m(y) = \emptyset$*

Proof. Suppose that X is Hausdorff, and let $x \neq y \in X$. There exist disjoint $u, v \in \mathfrak{T}$ with $x \in u, y \in v$. Then $m(x) \cap m(y) \subseteq {}^*u \cap {}^*v = {}^*(u \cap v) = \emptyset$.

Conversely, suppose $m(x) \cap m(y) = \emptyset$. Let u (respectively, v) be infinitesimal neighborhoods of x (respectively, y). Then $u \cap v \subseteq m(x) \cap m(y) = \emptyset$, that is,

$$(\exists u, v \in {}^*\mathfrak{T})[(x \in u) \wedge (y \in v) \wedge (u \cap v = \emptyset)]$$

The result now follows by transfer. $\dashv$

Direction (2) implies (1) in the last lemma can also be proved directly, using saturation. Suppose for a contradiction that X is not Hausdorff, so some $x \neq y$ cannot be separated by open sets. If $u_1, \ldots, u_n$ (respectively $v_1, \ldots, v_m$) are open neighborhoods of x (respectively, y) then

When the topology is Hausdorff, every $q \in NS({}^*X)$ is in $m(p)$ for exactly one p, which we call the *standard part* of q and denote by $\mathrm{st}(q)$ or ${}^\circ q$. In particular, if $q \in X$ then ${}^\circ q = q$. If $p, q \in NS({}^*X)$ are in the same monad, that is, when ${}^\circ p = {}^\circ q$, then write $p \approx q$ and say that p *is infinitely close to* q.

Exercise 1.1. *For each of the following topologies, describe the monads and determine whether the topology is Hausdorff.*

1. $X = \mathbb{N}, \mathfrak{T} = \mathcal{P}(\mathbb{N})$

2. $X = \mathbb{N}, \mathfrak{T} = \{A \subseteq \mathbb{N} : A^{\complement}$ *is finite*$\} \cup \{\emptyset\}$

3. $X = \mathbb{R}, \mathfrak{T}$ *is the* Sorgenfrey topology *generated by sets of the form* $[a, b), a < b \in \mathbb{R}$

4. $X = \mathbb{R}, \mathfrak{T}$ *is the topology generated by sets of the form* $[a, b), a < b, a, b \in \mathbb{Q}$

5. $X = \mathbb{R}, \mathfrak{T}$ *is the topology generated by sets of the form* $[a, b), a < b, a, b \in [-\infty, \infty]$

Theorem 1.1. *Let* $(X, \mathfrak{T})$ *be a topological space.*

1. *A set* $U \subseteq X$ *is open if and only if for every* $x \in U, m(x) \subseteq {}^*U$.

2. *A set* $E \subseteq X$ *is closed if and only if for every* $x \in E^{\complement}, m(x) \cap {}^*E = \emptyset$. *Equivalently,* E *is closed if for every* $x \in {}^*E$, *if* $x \in m(y)$ *then* $y \in E$.

3. *A set* $K \subseteq X$ *is compact if and only if every point in* ${}^*K$ *is nearstandard in* K, *that is, in the monad of a point of* K.

If the space is Hausdorff, then these become:

(1'.) U is open if and only if $\mathrm{st}^{-1} U \subseteq {}^*U$.

(2'.) E is closed if and only ${}^*E \cap NS({}^*X) \subseteq \mathrm{st}^{-1} E$.

(3'.) K is compact if and only if ${}^*K \subseteq \mathrm{st}^{-1} K$.

Equivalence 3 of Theorem 1.1 is the *Robinson Compactness Criterion.*

Proof. (1) If U is open and $x \in U$ then $m(x) \subseteq {}^*U$ by definition of $m(x)$. Conversely, suppose that $a \in m(a) \subseteq {}^*U$. Let u be an infinitesimal neighborhood of x, so $(\exists u \in {}^*\mathfrak{T})[x \in u \in {}^*U]$. By transfer, for every $a \in U$ there is an open u_a with $a \in u_a \subseteq U$, and $U = \bigcup_{a \in U} u_a$ is open.

(2) follows immediately from (1).

(3) Recall that a set K is compact provided every cover has an open subcover. Suppose first that K is compact but some $x \in {}^*K$ is not nearstandard in K. Then for every $a \in K$ $x \notin m(a)$, so there is an open u_a with $a \in u_a$ and $x \notin {}^*u_a$. As $\{u_a : a \in K\}$ is an open cover of K, it has a finite subcover $\{u_{a_1}, \ldots, u_{a_n}\}$, and by transfer $x \in {}^*K \subseteq {}^*u_{a_1} \cap \cdots \cap {}^*u_{a_n}$, a contradiction.

Conversely, let $\{u_i\}_{i\in I}$ be open sets with $K \subseteq \bigcup_{i\in I} u_i$, and suppose there is no finite subcover of K. Put $E_i = K \setminus u_i$, then $\{E_i\}_{i\in I}$ has the finite intersection property, and by transfer so does $\{{}^*E_i\}_{i\in I}$. By saturation there is some x in every ${}^*E_i$; that is, $x \in {}^*K$ and for all $i \in I, x \notin {}^*u_i$. Since for every $y \in K, m(y) \subseteq {}^*u_i$ for some i, x is not nearstandard in K. ⊣

Exercise 1.2. *Complete the following alternate proof that if every element of* ${}^*K$ *is nearstandard in* K *then* K *is compact. Given an open cover* $\mathfrak{U}$ *of* K*, there is a hyperfinite set* $\hat{\mathfrak{U}} \subseteq {}^*\mathfrak{T}$ *such that* ${}^*u \in \hat{\mathfrak{U}}$ *for every* $u \in \mathfrak{U}$*. Use the condition on* ${}^*K$ *to show that* ${}^*K \subseteq \bigcup \hat{\mathfrak{U}}$*. By transfer,* K *has a finite subcover from* $\mathfrak{U}$*.*

Exercise 1.3. *(a) Let* K *be a compact subset of a topological space* $X, \mathfrak{T}$*, then* $NS({}^*K) = \bigcap\{{}^*u : K \subseteq u \in \mathfrak{T}\}$*, and there is a* $\hat{u} \in {}^*\mathfrak{T}$ *with* ${}^*K \subseteq \hat{u} \subseteq NS(\{{}^*K\})$*. (b) If* K *and* L *are disjoint compact subsets of a Hausdorff space* $(X, \mathfrak{T})$*, then there are disjoint open sets* u *and* v *with* $K \subseteq u, L \subseteq v$*. [Hint for (b): Otherwise, use (a) and saturation to find an* $x \in \mathrm{st}^{-1}(K) \cap \mathrm{st}^{-1}(L)$*]*

Exercise 1.4. *Let* $X = \mathbb{R}$ *with the standard topology. Use the compactness criterion from Theorem 1.1 to prove that the set* $K = \{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty}$ *is compact in the standard topology, but not in the Sorgenfrey topology of Exercise 1.1.*

The following useful result is due to Luxemburg.

Theorem 1.2. *Let* $(X, \mathfrak{T})$ *be a Hausdorff topological space, and* A *an internal subset of* ${}^*X$*.*

1. $\mathrm{st}(A) = \{{}^\circ a : a \in A \cap NS({}^*X)\}$ *is closed*

2. *Suppose* $\mathfrak{T}$ *is* regular*; that is, whenever* E *is a closed subset of* X *and* $p \notin E$*, there exist disjoint open sets* u *and* v *with* $E \subseteq u, p \in v$*. Suppose* A *is an internal subset of* $NS({}^*X)$*. Then* $\mathrm{st}(A)$ *is compact.*

Proof. For (1), let $x \in X \setminus \mathrm{st}(A)$. it suffices to find a $u \in \mathfrak{T}$ with $x \in u \subseteq X \setminus \mathrm{st}(A)$. If not, then for every $u \in \mathfrak{T}$ with $x \in u$ there is an $a \in A$ with ${}^\circ a \in u$. It follows that $a \in {}^*u$. By saturation there is an a with $a \in A \cap {}^*u$ for every such u, so $a \in m(x)$, contradicting the assumption that $x \notin \mathrm{st}(A)$.

For (2), let $\mathfrak{U}$ be an open cover of $\mathrm{st}(A)$. For every $a \in A$ there is a $u_a \in \mathfrak{U}$ with ${}^\circ a \in u_a$. By regularity there are disjoint open v_a, w_a with ${}^\circ a \in v_a, u_a^{\complement} \subseteq w_a$. The set $\mathcal{V} = \{{}^*v_a : a \in A\}$ covers A, and even though it is indexed by A it is actually no larger than the standard set $\mathfrak{T}$, so saturation applies, and by Exercise **??** there is a finite subcover ${}^*v_{a_1}, \ldots, {}^*v_{a_n}$ from $\mathcal{V}$ of A. It remains to show that $u_{a_1}, \ldots, u_{a_n}$ is a finite subcover of $\mathrm{st}(A)$ from $\mathcal{U}$. If $a \in A$ then $a \in {}^*v_{a_i}$ for some i. Suppose (for a contradiction) that ${}^\circ a \notin u_{a_i}$. Then ${}^\circ a \in w_a$, so $a \in {}^*w_{a_i}$. Therefore ${}^*v_{a_i} \cap {}^*w_{a_i} \neq \emptyset$, which by transfer and the choice of v_a and w_a is a contradiction. ⊣

We conclude this section with applications of the nonstandard characterizations from Theorem 1.1

Theorem 1.3. *1. A closed subset of a compact set is compact.*

2. A compact subset of a Hausdorff space is closed.

Proof. If $E \subseteq K$, E closed and K compact, let $x \in {}^*E$. Since $x \in {}^*K, x \in m(a)$ for some $a \in K$. If $a \notin E$ then $x \in m(a) \subseteq {}^*E^{\complement}$, a contradiction; it follows that every point of ${}^*E$ is nearstandard in E. For the second part, let $x \in K^{\complement}$, and suppose $m(x) \cap {}^*K \neq \emptyset$. Let $y \approx x, y \in {}^*K$. Then $x = {}^\circ y \in K$ since K is compact. This contradicts the choice of x. ⊣

Exercise 1.5. *Let E be a subset of a topological space X. The following are equivalent:*

1. *x is a limit point of E, that is, every open neighborhood of x has nonempty intersection with E.*
2. *$m(x) \cap {}^*E \neq \emptyset$.*

Conclude that E is closed if and only if it contains all of its limit points.

2 Continuity

Theorem 2.1. *Let f be a function from one topological space $(X, \mathfrak{T})$ to another $(Y, \mathfrak{S})$. The following are equivalent:*

1. *f is* continuous*, that is, $f^{-1}(u) \in \mathfrak{T}$ for every $u \in \mathfrak{S}$*
2. *For every $x \in X$, ${}^*f(m_X(x)) \subseteq m_Y(f(x))$*

Proof. $(1 \Rightarrow 2)$Let $y \in m_X(x)$ and $u \in \mathfrak{S}$ with $f(x) \in u$; then $m_X(x) \subseteq {}^*f^{-1}(u)$, so ${}^*f(y) \in {}^*u$. Since u was arbitrary, ${}^*f(y) \in m_Y(f(x))$.

$(2 \Rightarrow 1)$ If $u \in \mathfrak{S}$ and $x \in f^{-1}(u)$ then $f(m_X(x)) \subseteq m_Y(f(x)) \subseteq {}^*u$ so $m_X(x) \subseteq {}^*f^{-1}(u)$; by Theorem 1.1, $f^{-1}(u) \in \mathfrak{T}$. ⊣

For a Hausdorff space, this theorem becomes:

f is continuous if and only if for all $x \in X$ and $y \approx x, {}^*f(y) \approx {}^*f(x)$

Exercise 2.1. *(Continuity at a point) Show that for any $x \in X$, ${}^*f(m_X(x)) \subseteq m_Y(f(x))$ if and only if for every neighborhood u of $f(x)$ there is a neighborhood v of x with $f(v) \subseteq u$.*

Exercise 2.2. *Use Robinson's compactness criterion to show that if $f : X \to Y$ is continuous and K is a compact subset of X then $f(K)$ is a compact subset of Y.*

Exercise 2.3. *Let f be a continuous real-valued function on a Hausdorff topological space X and $K \subseteq X$ be compact. Show there is a hyperfinite subset Ω of ${}^*K$ such that $K = \operatorname{st}\Omega$. Note that ${}^*f$ has a maximum value at some $\omega \in \Omega$ among all elements of Ω. Conclude that f attains a maximum on K at ${}^\circ\omega$.*

Exercise 2.4. *Let X be the set of bounded continuous real functions on $\mathbb{R}$. Let d be the* sup *metric on X (see Section XXXX), that is, $d(f,g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)|$. This generates a topology on X with basic open sets of the form $\{g \in X : d(f,g) < \epsilon\}$ with $f \in X$ and $\epsilon > 0$. Show that X is not compact. (Hint: consider the function ${}^*\sin(Hx)$ for H infinite.)*

3 Applications

In this section we give a few applications which demonstrate the power of the nonstandard methods. These are results of the sort one normally encounters in an advanced undergraduate or beginning graduate course in Analysis or Topology, and we invite the reader to compare the proofs here to those in conventional textbooks in these fields.

Theorem 3.1. *Let $(X, <)$ be a linearly-ordered set, with the* order topology *generated by intervals of the form $(a, b) = \{x \in X : a < x < b\}$ where $a, b \in X$. Suppose that X satisfies the* upper bound property*; that is, every subset of X which is bounded above has a least upper bound. Then every interval of the form $[a, b] = \{x \in X : a \le x \le b\}$ is compact.*

Compare to Theorem 6.1 of Munkres [?].

Proof. Let $x \in {}^*[a,b]$, and put $x' = \sup E$ where $E = \{y \in X : y^* \leq x\}$. Since $a \in E, a \leq x'$. Since b is an upper bound for E, $x' \leq b$. By the Robinson Compactness Criterion, it remains to show that $x \in m(x')$. Suppose not; then there exist $\alpha, \beta \in X$ with $a < \alpha < x' < \beta < b$ with $x \notin {}^*(\alpha, \beta)$. However, if $x^* > \beta$ then $\beta \in E$ and $x' \geq \beta$, a contradiction. Similarly, if $x^* < \alpha$ then α is an upper bound of E, and $x' \leq \alpha$, also a contradiction. Therefore, $x \in m(x')$, and the result follows. ⊣

Note that the x' defined in the above proof is the standard part of x. This is quite common in nonstandard proofs in topology: rather than take the standard part of x and show that it satisfies our requirements, we *constructed* a point x' satisfying our requirements and showed that it was the standard part of x.

The last result generalizes a special case of the *Heine-Borel Theorem*[**?**]:

Theorem 3.2. *A subset K of $\mathbb{R}^n$ is compact if and only if K is closed and bounded.*

We recall that the standard topology on $\mathbb{R}^n$ is the one generated by basic open sets of the form $u_1 \times \cdots \times u_n$, where each u_i is an open interval in $\mathbb{R}$, and that $\mathbb{R}^n$ is Hausdorff (as is any metric space; see Chapter **??**). The reader should confirm that for any $\mathbf{x} \in \mathbb{R}^n, m(\mathbf{x}) = \{\mathbf{y} \in {}^*\mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| \approx 0\} = m(x_1) \times \cdots \times m(x_n)$.

Proof. ($\Rightarrow$) If K is compact then K is closed by Theorem 1.3. If K is unbounded then by saturation there is an $\mathbf{x} \in {}^*K$ with $\|\mathbf{x}\|$ infinite; but $\|\mathbf{x}\| \leq \|\mathbf{x} - \operatorname{st}\mathbf{x}\| + \|\operatorname{st}\mathbf{x}\| \approx \|\operatorname{st}\mathbf{x}\|$, which is finite. ($\Leftarrow$) Suppose K is closed and bounded, and let $\mathbf{x} = \langle x_1, \ldots, x_n \rangle \in {}^*\mathbb{R}^n$. Since $|x_i| \leq \|\mathbf{x}\|$ is finite, it has a standard part in $\mathbb{R}$, and ${}^\circ\mathbf{x} = \langle {}^\circ x_1, \ldots, {}^\circ x_n \rangle$, which is in K since K is closed, by Theorem 1.1. ⊣

The next result is often called the *Sunrise Lemma*.

Theorem 3.3 (F. Riesz)**.** *Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$, and suppose $\lim_{x \to \infty} f(x) = -\infty, \lim_{x \to -\infty} f(x) = \infty$. Then $G = \{x \in \mathbb{R} : (\exists y > x)[f(y) > f(x)]\}$ is open, and if (a,b) is a bounded component of G then $f(a) = f(b)$*

Proof. If $x \in G$ and $z \approx x$ then for some standard $y > x, f(y) > f(x)$. Since $y - x$ and $f(y) - f(x)$ are standard positive numbers, and $f(x) \approx {}^*f(z)$,

$y^* > z$ and $f(y)^* >^* f(z)$, so $z \in {}^*G$, proving that G is open. Now, suppose that (a, b) is a component of G. Since $a, b \notin G, f(b) \geq f(y)$ for all $y \geq b$, and $f(a) \geq f(y)$ for all $y \geq a$; in particular, $f(a) \geq f(b)$. Suppose (for a contradiction) that $f(a) > f(b)$. Let $a' \approx a, a'^* > a$. Let c be the element of the internal interval $[a', b]$ at which ${}^*f$ is maximized. Then ${}^*f(c) \geq {}^*f(a') \approx f(a) > f(b)$, so $c \neq b$; it follows that $c \in {}^*G$ and for some $y > c, {}^*f(y) > {}^*f(c)$. By choice of c, $y^* > b$, but then $f(b) \geq {}^*f(y) > {}^*f(c) \geq {}^*f(a') \approx f(a) > f(b)$, a contradiction. ⊣

For a standard proof of the next result see N. Fine, *Amer. Math. Monthly*, vol. 73, 1966, p. 782.

Theorem 3.4. *let $f : \mathbb{R} \to \mathbb{R}$, and suppose*

1. *f has the* intermediate value property*, that is, whenever $f(a) < r < f(b)$ then $f(c) = r$ for some $c \in (a, b)$; and*

2. *For all $r \in \mathbb{Q}$, $f^{-1}(r)$ is a closed subset of $\mathbb{R}$.*

Then f is continuous.

Proof. Otherwise suppose $x \in \mathbb{R}$, $z \approx x$, and ${}^*f(z) \not\approx f(x)$. For definiteness, assume $x < z$ and $f(x) < {}^*f(z)$. There is a standard rational r with $f(x) < r < {}^*f(z)$. By the transfer of the intermediate value property, there is a $c \in {}^*\mathbb{R}$ with $x < c < z$ and ${}^*f(c) = r$. Then $c \in m(x) \cap {}^*f^{-1}(r)$, and by closure of $f^{-1}(r)$ and Theorem 1.1, $f(x) = r$, a contradiction. ⊣

A *topological vector space* is a (real or complex) vector space X on which there is a Hausdorff topology $\mathfrak{T}$ with respect to which addition and scalar multiplication are continuous on $X \times X$ and $\mathbb{R} \times X$ (or $\mathbb{C} \times X$). The following result, for $X = \mathbb{R}^n$, is [**?**], Chapter 4, Exercise 25.

Theorem 3.5. *Let X be a topological vector space, $C \subseteq X$ be closed, and $K \subseteq X$ be compact. Then $K + C = \{k + c : k \in K, c \in C\}$ is closed in X.*

Proof. Let $x \in {}^*(K + C)$ be nearstandard, it suffices to show that $\mathrm{st}(x) \in K + C$. $x = k + c$ for some $k \in {}^*K$ and $c \in {}^*C$. k is nearstandard with ${}^\circ k \in K$, so $c = x - k$ is nearstandard and ${}^\circ c = {}^\circ x - {}^\circ k \in C$ since C is closed. Then ${}^\circ x = {}^\circ k + {}^\circ c \in K + C$, as desired. ⊣

Now, suppose that $(X_i, \mathfrak{T}_i)$ are topological spaces for $i \in I$. The *product topology* $\mathfrak{T}$ on $X = \prod_i X_i$ is the topology generated by basic open sets of the form $\prod_i u_i$, where each $u_i \in \mathfrak{T}_i$ and $u_i = X_i$ for all but finitely many $i \in I$.

Theorem 3.6 (Tychonoff)**.** *Suppose X_i is compact for all i. Then $\prod_i X_i$ is compact in the product topology.*

Proof. Let $\mathbf{x} = \langle x_i \rangle_{i \in {}^*I} \in {}^*X$. Since X_i is compact, for each standard $i \in I$ there is a $y_i \in X_i$ with $x_i \in m_{X_i}(y_i)$. Put $\mathbf{y} = \langle y_i \rangle_{i \in I}$. It suffices to show that $\mathbf{x} \in m(\mathbf{y})$.

Suppose $y_i \in u_i \in \mathfrak{T}_i$ and $I' = \{i \in I : u_i \neq X_i\}$ is finite, say $I' = \{i_1, \ldots, i_n\}$, and put $u = \prod_{i \in I} u_i$. By transfer of the statement

$$(\forall i \in I)[((i \neq i_1) \wedge (i \neq i_2) \wedge \cdots \wedge (i \neq i_n)) \Rightarrow (u_i = X_i)]$$

we see that ${}^*u = \prod_{i \in {}^*I} v_i$ where $v_i = \begin{cases} {}^*u_i & \text{if } i \in I' \\ {}^*X_i & \text{if } i \in {}^*I, i \notin I'. \end{cases}$ Then $\mathbf{x} \in {}^*u$. Since u was an arbitrary basic open neighborhood of $\mathbf{y}$, $\mathbf{x} \in m(\mathbf{y})$. ⊣

The proof of Theorem 3.6 characterizes the monads in the product topology, namely, if $\mathbf{x} = \langle x_i \rangle_{i \in I} \in X$ then $m(\mathbf{x}) = \prod_{i \in {}^*I} E_i$ where $E_i = \begin{cases} m(x_i) & \text{if } i \in I \\ {}^*X_i & \text{if } i \in {}^*I \setminus I. \end{cases}$

Exercise 3.1. *Suppose that $(X_i, \mathfrak{T}_i)$ are topological spaces for $i \in I$. The* box topology $\mathfrak{T}$ *on $X = \prod_i X_i$ is the topology generated by basic open sets of the form $\prod_i u_i$, where each $u_i \in \mathfrak{T}_i$. Describe the monads under the box topology.*

Note that in the case where the spaces X_i are Hausdorff, if $\mathbf{x} = \langle x_i \rangle_{i \in {}^*I} \in {}^*X$ is nearstandard in ${}^*X$ then the standard part of $\mathbf{x}$ is $\langle {}^\circ x_i \rangle_{i \in I}$. (This is true in both the product and box topologies!)

We conclude this section with a version of the Closed Graph Theorem.

Theorem 3.7. *Let f be a function from one topological space $(X, \mathfrak{T})$ to another $(Y, \mathfrak{S})$. Let $\Gamma(f)$ be the graph of f in $X \times Y$.*

1. *If f is continuous and Y is Hausdorff then $\Gamma(f)$ is closed.*

2. *If $\Gamma(f)$ is closed and Y is compact then f is continuous.*

Proof. 1. Suppose $(x, y) \notin \Gamma(f)$ and $(x', y') \approx (x, y)$. ${}^*f(x') \approx f(x) \not\approx y \approx y'$, so $y' \neq {}^*f(x')$, that is, $(x', y') \notin {}^*\Gamma(f)$.

2. Let $x \in X$ and $x' \in m(x)$. Since Y is compact, ${}^*f(x')$ is nearstandard, so $(x', {}^*f(x'))$ is nearstandard, in fact in $m_{X\times Y}((x, \mathrm{st}({}^*f(x'))))$. Also $(x', {}^*f(x')) \in {}^*\Gamma(f)$, and $\Gamma(f)$ is closed, so $(x, \mathrm{st}({}^*f(x'))) \in \Gamma(f)$, so ${}^*f(x') \in m(f(x))$, proving continuity.

$\dashv$

4 S-topology and the Stone-Čech Compactification

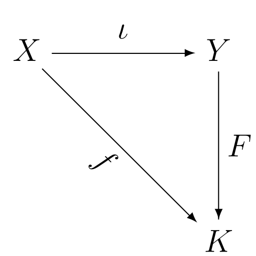
Suppose $(X, \mathfrak{T})$ is a topological space. The external set $\mathcal{B} = \{{}^*U : U \in \mathfrak{T}\}$ forms the basis for a topology $\mathfrak{T}_S$ on ${}^*X$, the *S-Topology* built from $(X, \mathfrak{T})$.

Theorem 4.1. *If* $\mathrm{card}(\mathcal{B}) < \kappa$ *then* ${}^*X$ *is compact in the S-topology.*

Proof. Let $\{V_i\}_{i\in I}$ be an open cover of ${}^*X$. Every V_i the union of the basic elements contained in it, $V_i = \cup W_i$ where $W_i = \{U \in \mathcal{B} | U \subseteq V_i\}\}$. Then $\bigcup_{i\in I} W_i$ is an open cover of ${}^*X$ by internal sets, and even if $card(I)$ is large the cardinality of $\bigcup_{i\in I} W_i$ is less than κ. It follows from saturation that there is a finite subcover of ${}^*X$ from $\bigcup_{i\in I} W_i$, and therefore from $\{V_i\}_{i\in I}$. $\dashv$

We now use this S-topology to construct the *Stone-Čech compactification.*

Theorem 4.2. *Let* $(X, \mathfrak{T})$ *be a topological space. There is a compact topological space* Y *and a map* $\iota : X \to Y$ *such that for any compact Hausdorff space* K *and continuous function* $f : X \to Y$ *there is a unique continuous function* $F : Y \to K$ *such that the map*



commutes.

Proof. Endow $^*X$ with the S-topology, let Y be the (compact) closure of X in $^*X$, and let ι be the natural embedding of X into $^*X$. Suppose K and f are as in the hypothesis. For $x \in Y$ let $F(x) =^{\circ *} f(x)$; this is well-defined since K is compact Hausdorff. The diagram commutes since for $x \in X$, $f(x) = {}^*f(x) =^{\circ *} f(x)$.

To prove that F is continuous it suffices to prove that for any open subset v of K,

$$F^{-1}(v) = Y \cap \bigcup \{ {}^*f^{-1}(w) : w \text{ is open}, \overline{w} \subseteq v \}$$

since the right-hand side is open in the S-topology restricted to Y.If $x \in Y$ with $z = F(x) \in v$, by Exercise 1.3 there is an open w with $z \in \overline{w} \subseteq v$. Since w is open, ${}^*f(x) \in {}^*w$. This proves $\subseteq$.

To prove the opposite inclusion, if $x \in Y$, w open with $\overline{w} \subseteq v$ and ${}^*f(x) \in {}^*w$ then $F(x) =^{\circ *} f(x) \in \overline{w} \subseteq v$.

It remains to prove uniqueness of F. Suppose $G : Y \to K$ is another continuous function such that $G(x) = f(x)$ for $x \in X$ but such that $G(y) \neq F(y)$ for some $y \in Y$. By continuity and the definition of the S-topology, there is a $u \in \mathfrak{T}$ and standard $\epsilon > 0$ with $y \in {}^*u$ and $|G - F| > \epsilon$ on ${}^*u$. As y is in the closure of X, ${}^*u$ contains some $x \in X$, but then $\epsilon < |G(x) - F(x)| = |G(x) - f(x)|$, a contradiction. $\dashv$

STONE-WEIERSTRASS?

5 Notes

Theorem