Math 411 **Final Exam** Due May 12, 2014
Professor: Tom Craven

Instructions: Write your answers clearly and completely with reasons for all your statements. The number in parentheses is the number of points the problem is worth. Return all exams to my office by 2PM on the due date. You are welcome to use your text, the online text, a calculus book and your notes, nothing else. You can also ask me questions, but not other people. Solutions will be posted online after the final.

(20) **1.** Let $V = M \oplus N$ be a finite dimensional vector space and assume that $E\colon V \to V$ is a projection on M along N. (That is, $E^2 = E$, the range of E is M and the kernel of E is N.) Show that the transpose $E^t\colon V^* \to V^*$ is a projection on N° along M°.

(25) **2.** Find the characteristic and minimal polynomials of

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}.$$

(30) **3.** We know that the Gram-Schmidt process will always provide an orthonormal basis for a finite dimensional inner product space V. But these bases are far from unique. For example, for $\mathbb{R}^2$, any rotation of the standard basis gives another orthonormal basis. For spaces of polynomials, things are much nicer. It is clear that if we start with the basis $\{1, x, x^2, \ldots, x^n\}$, then the Gram-Schmidt process gives an orthonormal set of polynomials $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ where $\deg p_k(x) = k$. You will show that these are very close to being unique. Let V be the vector space of polynomials over $\mathbb{R}$ of degree at most n with some inner product (f, g). (It may involve either integrals or discrete computations at points as in Lagrange interpolation – we don't need to know how it is defined.)

a. Let $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ be the orthonormal set of polynomials coming from the Gram-Schmidt process and let $p(x)$ be any polynomial of degree less than $k \le n$. Show that $(p(x), p_i(x)) = 0$ for $i \ge k$. [Hint: think of how $p(x)$ is expressed in terms of the orthogonal basis.]

b. Let $\{p_j(x)\}_{j=0}^n$ be as above. Let $\{q_j(x)\}_{j=0}^n$ be any other set of orthonormal polynomials with respect to the inner product that also satisfies $\deg q_k(x) = k$, $k = 0, 1, \ldots, n$. Prove that $q_j(x) = \pm p_j(x)$ for each $j = 0, 1, \ldots, n$. [Hint: write $p_k(x)$ in terms of the $q_j(x)$ basis and use part (a).]

(20) **4.** Let A be a matrix with characteristic polynomial $(x+1)^3(x-2)$. Give all possible minimal polynomials for A and for each one, give all possible Jordan canonical forms. How can you tell which is correct from the one number $\operatorname{rank}(A + I)$?

(20) **5.** Let $V = \mathbb{C}^2$. Define a linear transformation $T\colon V \to V$ by $T(z, w) = (z + w, iz - 2w)$.

a. Find the matrix of T with respect to the standard basis.

b. Find the matrix of the classical adjoint (adjugate) of T.

c. Find the product of the matrices in (a) and (b). [Hint: this gives you a check on your work.]

d. Find the matrix of the (hermitian) adjoint linear transformation T^*.

(30) **6.** Let V be the real inner product space consisting of all continuous real-valued functions on $[-1, 1]$ with inner product defined by

$$(f, g) = \int_{-1}^{1} f(t)g(t)\,dt.$$

Let W be the subspace of all odd functions in V (i.e. $f(-t) = -f(t)$).

a. Show that the subspace U of all even functions is orthogonal to W. [Hint: review calculus if necessary; you may use a calculus book.]

b. Show that $V = U \oplus W$ and $U = W^{\perp}$. [Careful: our earlier theorem does not apply since W is not finite dimensional.]

(25) **7.** Let V be a finite dimensional vector space and let $T \in L(V, V)$ have only the eigenvalues 2 and 3. Let W be the subspace of V on which $Tw = 2w$ and let U be the subspace of V on which $Tu = 3u$.

a. Assume that $\dim U + \dim W = \dim V$. Show that $V = U \oplus W$.

b. Give an example to show that (a) is not true without the assumption $\dim U + \dim W = \dim V$. [Hint: what might be the Jordan canonical form of the matrix for T?]

(30) **8.** We begin by working with the vector space $P(\mathbb{R})$ of all polynomial functions over the real numbers and consider two linear transformations from $P(\mathbb{R})$ to itself:

$$D\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=1}^{n} i c_i x^{i-1}$$

$$T\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} \frac{c_i}{1+i} x^{i+1}.$$

From calculus and work this semester, we know that T is nonsingular, but not invertible, D has kernel generated by the constant functions, $DT = I$ and $TD \neq I$.

a. State the relevant theorem from calculus to justify $T[(Tf)g] = (Tf)(Tg) - T[f(Tg)]$, for all $f, g \in P(\mathbb{R})$. Do the same for a similar rule for D.

b. Suppose that V is a nonzero subspace of $P(\mathbb{R})$ such that Tf belongs to V for each $f \in V$ (that is, V is invariant under T). Show that V is not finite-dimensional.

c. Suppose V is a finite-dimensional subspace of $P(\mathbb{R})$. Prove that there is an integer $m \geq 0$ such that $D^m f = 0$ for all $f \in V$. What does this tell you about the eigenvalues of D (as a linear transformation on all of $P(\mathbb{R})$)?

d. Expand the vector space from $P(\mathbb{R})$ to the space of all real analytic functions (real-valued functions on $\mathbb{R}$ having continuous derivatives of all orders). D is still differentiation and $T(f) = \int_0^x f(t)\,dt$. Everything above still holds except for the conclusion on eigenvalues. Find a nonzero eigenvalue for D. [Hint: given any real number λ, you should be able to find an associated eigenvector (usually called an *eigenfunction* in this case).]