Math 411 **Final Exam** Due May 12, 2014
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Instructions: Write your answers clearly and completely with reasons for all your statements. The number in parentheses is the number of points the problem is worth. Return all exams to my office by 2PM on the due date. You are welcome to use your text, the online text, a calculus book and your notes, nothing else. You can also ask me questions, but not other people. Solutions will be posted online after the final.

(20) **1.** Let $V = M \oplus N$ be a finite dimensional vector space and assume that $E\colon V \to V$ is a projection on M along N. (That is, $E^2 = E$, the range of E is M and the kernel of E is N.) Show that the transpose $E^t\colon V^* \to V^*$ is a projection on N° along M°.

Let $f \in V^*$. We know that $V^* = M^\circ \oplus N^\circ$, so we can write $f = g + h$ with $g \in M^\circ$ and $h \in N^\circ$. By definition, $E^t f(v) = f(E(v))$ for any $v \in V$. Writing $v = m + n$, with $m \in M$ and $n \in N$, we have $E^t f(v) = f(E(v)) = f(m)$. If $f \in N^\circ$, then $E^t f(v) = f(m) = f(v)$, so $E^t f = f$. And if $f \in M^\circ$, then $E^t f(v) = f(m) = 0$, so $E^t f = 0$. This shows that $E^t\colon V^* \to V^*$ is a projection on N° along M°.

(25) **2.** Find the characteristic and minimal polynomials of

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}$$

$f_A(x) = \det(xI - A) = (x+4)(x-3)^2$. The minimal polynomial is the same since $(A+4I)(A-3I) \neq 0$ and this is the only other polynomial that has the same eigenvalues to lower degrees.

(30) **3.** We know that the Gram-Schmidt process will always provide an orthonormal basis for a finite dimensional inner product space V. But these bases are far from unique. For example, for $\mathbb{R}^2$, any rotation of the standard basis gives another orthonormal basis. For spaces of polynomials, things are much nicer. It is clear that if we start with the basis $\{1, x, x^2, \ldots, x^n\}$, then the Gram-Schmidt process gives an orthonormal set of polynomials $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ where $\deg p_k(x) = k$. You will show that these are very close to being unique. Let V be the vector space of polynomials over $\mathbb{R}$ of degree at most n with some inner product (f, g). (It may involve either integrals or discrete computations at points as in Lagrange interpolation – we don't need to know how it is defined.)

a. Let $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ be the orthonormal set of polynomials coming from the Gram-Schmidt process and let $p(x)$ be any polynomial of degree less than $k \leq n$. Show that $(p(x), p_i(x)) = 0$ for $i \geq k$. [Hint: think of how $p(x)$ is expressed in terms of the orthogonal basis.]

Since the basis is orthonormal, $p(x) = \sum_{i=0}^{n} (p(x), p_i(x)) p_i(x)$. If $k \leq n$, then $\deg p < n$, so the coefficient $(p(x), p_n(x))$ must be zero (otherwise the degree on the right hand side will be n). Induction downward gives $(p(x), p_i(x)) = 0$ for each i greater than the degree of p, so for $i \geq k > \deg p$.

b. Let $\{p_j(x)\}_{j=0}^n$ be as above. Let $\{q_j(x)\}_{j=0}^n$ be any other set of orthonormal polynomials with respect to the inner product that also satisfies $\deg q_k(x) = k$, $k = 0, 1, \ldots, n$. Prove that $q_j(x) = \pm p_j(x)$ for each $j = 0, 1, \ldots, n$. [Hint: write $p_k(x)$ in terms of the $q_j(x)$ basis and use part (a).]

By part (a), $(p_k(x), q_i(x)) = (q_i(x), p_k(x)) = 0$ for $k > i$. For each k, we can write $p_k(x) = \sum_{i=k}^n (p_k(x), q_i(x)) q_i(x)$. A degree argument like (a) shows that $(p_k(x), q_i(x)) = 0$ for $i > k$. Thus $p_k(x) = (p_k(x), q_k(x)) q_k(x)$. Taking the inner product with $p_k(x)$, gives $1 = (p_k(x), p_k(x)) = (p_k(x), q_k(x))^2$, so that $(p_k(x), q_k(x)) = \pm 1$.

(20) **4.** Let A be a matrix with characteristic polynomial $(x+1)^3(x-2)$. Give all possible minimal polynomials for A and for each one, give all possible Jordan canonical forms. How can you tell which is correct from the one number $\operatorname{rank}(A+I)$?

The minimal polynomial divides the characteristic polynomial and has the same roots. Thus the possibilities are $(x+1)(x-2)$, $(x+1)^2(x-2)$ and $(x+1)^3(x-2)$. The corresponding matrices are

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Since the rank of $A + I$ is the same as the rank of the Jordan form, we see that these matrices added to I are distinguished by having ranks of 1, 2 and 3, respectively.

(20) **5.** Let $V = \mathbb{C}^2$. Define a linear transformation $T \colon V \to V$ by $T(z, w) = (z + w, iz - 2w)$.
a. Find the matrix of T with respect to the standard basis.

$$\begin{pmatrix} 1 & 1 \\ i & -2 \end{pmatrix}$$

b. Find the matrix of the classical adjoint (adjugate) of T.

$$\begin{pmatrix} -2 & -1 \\ -i & 1 \end{pmatrix}$$

c. Find the product of the matrices in (a) and (b). [Hint: this gives you a check on your work.]

$(\det A)I = (-2-i)I$

d. Find the matrix of the (hermitian) adjoint linear transformation T^*.

$$\begin{pmatrix} 1 & -i \\ 1 & -2 \end{pmatrix}$$

(30) **6.** Let V be the real inner product space consisting of all continuous real-valued functions on $[-1,1]$ with inner product defined by

$$(f,g) = \int_{-1}^{1} f(t)g(t)\,dt.$$

Let W be the subspace of all odd functions in V (i.e. $f(-t) = -f(t)$).

a. Show that the subspace U of all even functions is orthogonal to W. [Hint: review calculus if necessary.]

We need to show that if $f \in W$, $g \in U$, the inner product is 0. But

$$\begin{aligned}
(f,g) &= \int_{-1}^{1} f(t)g(t)\,dt = \int_{-1}^{0} f(t)g(t)\,dt + \int_{0}^{1} f(t)g(t)\,dt \\
&= -\int_{1}^{0} f(-u)g(-u)\,dt + \int_{0}^{1} f(t)g(t)\,dt \qquad \text{(substitute } u=-t\text{)} \\
&= \int_{0}^{1} f(-u)g(-u)\,dt + \int_{0}^{1} f(t)g(t)\,dt \\
&= -\int_{0}^{1} f(u)g(u)\,dt + \int_{0}^{1} f(t)g(t)\,dt = 0 \qquad \text{(using } f \text{ odd and } g \text{ even)}
\end{aligned}$$

b. Show that $V = U \oplus W$ and $U = W^{\perp}$. [Careful: our earlier theorem does not apply since W is not finite dimensional.]

We see that $V = U + W$ from the usual formula for writing a function as the sum of an even and odd function $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$. The intersection is zero since $f \in U \cap W$ implies that $f(-x) = f(x) = -f(x)$, so $f(x) = 0$ for every x. To see that $U = W^{\perp}$, let $f \in W^{\perp}$. f can be written uniquely as $g + h$ with $g \in U$ and $h \in W$. Now by (a), we have $(g,h) = 0$, so $0 = (f,h) = (g+h,h) = (g,h) + (h,h) = (h,h)$ implies that $h = 0$, hence $f = g \in U$.

(25) **7.** Let V be a finite dimensional vector space and let $T \in L(V, V)$ have only the eigenvalues 2 and 3. Let W be the subspace of V on which $Tw = 2w$ and let U be the subspace of V on which $Tu = 3u$.

a. Assume that $\dim U + \dim W = \dim V$. Show that $V = U \oplus W$.

Let $v \in U \cap W$. Then $Tv = 2v = 3v \implies v = 0$. Therefore, $U \oplus W$ is a subspace of V of dimension $\dim U + \dim W - \dim(U \cap W) = \dim U + \dim W = \dim V$, so $V = U \oplus W$.

b. Give an example to show that (a) is not true without the assumption $\dim U + \dim W = \dim V$. [Hint: what might be the Jordan canonical form of the matrix for T?]

Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by multiplication by the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Then $\dim U = 1 = \dim W$, so $U \oplus W$ is a 2-dimensional subspace of $\mathbb{R}^3$.

(30) **8.** We begin by working with the vector space $P(\mathbb{R})$ of all polynomial functions over the real numbers and consider two linear transformations from $P(\mathbb{R})$ to itself:

$$D\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=1}^{n} i c_i x^{i-1}$$
$$T\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} \frac{c_i}{1+i} x^{i+1}.$$

From calculus and work this semester, we know that T is nonsingular, but not invertible, D has kernel generated by the constant functions, $DT = I$ and $TD \neq I$.

a. State the relevant theorem from calculus to justify $T[(Tf)g] = (Tf)(Tg) - T[f(Tg)]$, for all $f, g \in P(\mathbb{R})$. Do the same for a similar rule for D.

The first is the integration by parts formula; The corresponding rule for differentiation is the product formula $D(fg) = fD(g) + gD(f)$.

b. Suppose that V is a nonzero subspace of $P(\mathbb{R})$ such that Tf belongs to V for each $f \in V$ (that is, V is invariant under T). Show that V is not finite-dimensional.

There exists a nonzero polynomial $f \in V$. The set $\{f, Tf, T^2 f, \dots\}$ is linearly independent since the polynomials have different degrees ($\deg T^k f = \deg f + k$), and so $\dim V = \infty$.

c. Suppose V is a finite-dimensional subspace of $P(\mathbb{R})$. Prove that there is an integer $m \geq 0$ such that $D^m f = 0$ for all $f \in V$. What does this tell you about the eigenvalues of D (as a linear transformation on all of $P(\mathbb{R})$)?

Let $f_1, \ldots, f_n$ be a basis for V. Let $m = \max \deg f_i + 1$. Then $D^m(\sum a_i f_i(x)) = \sum a_i f_i^{(m)}(x) = 0$ for any linear combination of basis elements. This tells us that the minimal polynomial of D on V is a power of x, and thus all the eigenvalues are 0. Since any eigenvector of D generates a 1-dimensional subspace of $P(\mathbb{R})$, the corresponding eigenvalue is 0, hence ALL eigenvalues of D are zero.

d. Expand the vector space from $P(\mathbb{R})$ to the space of all real analytic functions (real-valued functions on $\mathbb{R}$ having continuous derivatives of all orders). D is still differentiation and $T(f) = \int_0^x f(t)\,dt$. Everything above still holds except for the conclusion on eigenvalues. Find a nonzero eigenvalue for D. [Hint: given any real number λ, you should be able to find an associated eigenvector (usually called an *eigenfunction* in this case).]

This asks for an analytic function satisfying $D(f) = f'(x) = \lambda f(x)$. An obvious solution is $f(x) = e^{\lambda x}$.