A Quick Trip through Category Theory

Most topics here are standard to any graduate algebra book. The concept of universal objects for a functor is treated differently in most books and is largely ignored in general in [S] (though it is implicit in all the standard constructions: products, coproducts, free objects, etc.). These notes will follow [J2].

Definition. A *category* $\mathcal{C}$ consists of
(a) a class of *objects* $ob(\mathcal{C})$;
(b) for each pair of objects $A, B \in ob(\mathcal{C})$, a set $Mor_{\mathcal{C}}(A, B)$ of *morphisms* of A to B. These sets are all disjoint, in the sense that $Mor(A, B) \cap Mor(C, D) \neq \emptyset \implies A = C$ and $B = D$;
(c) for each triple of objects $A, B, C \in ob(\mathcal{C})$, there is a function called *composition*

$$Mor(B, C) \times Mor(A, B) \to Mor(A, C).$$

For $\beta \in Mor(B, C)$, $\alpha \in Mor(A, B)$, write $\beta\alpha$ or $\beta \circ \alpha$ for the image of (β, α). Composition satisfies

(1) **associative law:** $\gamma(\beta\alpha) = (\gamma\beta)\alpha$ for all $\alpha \in Mor(A, B), \beta \in Mor(B, C), \gamma \in Mor(C, D)$.

(2) **identity law:** for all $A \in ob(\mathcal{C})$, there exists $1_A \in Mor(A, A)$ called the *identity morphism* such that for any $B \in ob(\mathcal{C})$, $\beta 1_A = \beta$ and $1_A\gamma = \gamma$, $\forall \beta \in Mor(A, B)$, $\forall \gamma \in Mor(B, A)$.

Examples.
(1) $\mathcal{S}ets$, whose objects are sets and morphisms are set functions.
(2) $\mathcal{G}p$, whose objects are groups and morphisms are group homomorphisms.
(3) $\mathcal{T}op$, whose objects are topological spaces and morphisms are continuous functions.
(4) $\mathcal{A}b$, whose objects are abelian groups and morphisms are group homomorphisms.
(5) $\mathcal{A}$, where $ob(\mathcal{A}) = \{\mathcal{O}\}$ and $Mor(\mathcal{O}, \mathcal{O}) = \{0, 1, 2, 3, \ldots\}$ with composition of morphisms being addition.

We shall write $f : A \to B$ to mean $f \in Mor(A, B)$.
Define *subcategory* and *full subcategory*, *isomorphism, monomorphism, epimorphism.*

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A *covariant functor* $F : \mathcal{C} \to \mathcal{C}'$ is a pair of functions; one associates an object $F(A) \in ob(\mathcal{C}')$ for each object $A \in ob(\mathcal{C})$, and the other associates a morphism $F(f) : F(A) \to F(B)$ to each morphism $f : A \to B$ for $A, B \in ob(\mathcal{C})$, such that
(a) composition is preserved;
(b) identity is preserved.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{C}'$ is defined similarly, except that to each morphism $f : A \to B$ in $\mathcal{C}$, F assigns a morphism $F(f) : F(B) \to F(A)$ in $\mathcal{C}'$.

Examples.

(1) The covariant power set functor $\mathcal{P}\colon \mathcal{S}ets \to \mathcal{S}ets$: if $f\colon A \to B$, then $\mathcal{P}(f)\colon \mathcal{P}(A) \to \mathcal{P}(B)$ is defined by $\mathcal{P}(f)(X) = f(X)$ for any $X \subseteq A$.

(2) The *forgetful functor* $F\colon \mathcal{G}p \to \mathcal{S}ets$.

(3) For any category $\mathcal{C}$, the covariant hom functor: for $A \in ob(\mathcal{C})$, the functor $\hom_A\colon \mathcal{C} \to \mathcal{S}ets$ is defined by $\hom_A(C) = Mor(A, C)$ for all $C \in ob(\mathcal{C})$; if $f\colon C \to C'$ is a morphism, then $\hom_A(f)\colon Mor(A, C) \to Mor(A, C')$ is defined by $\phi \mapsto f \circ \phi$ for all $\phi \in Mor(A, C)$.

(4) For any category $\mathcal{C}$, the contravariant hom functor: for $A \in ob(\mathcal{C})$, the functor $\hom^A\colon \mathcal{C} \to \mathcal{S}ets$ is defined by $\hom^A(C) = Mor(C, A)$ for all $C \in ob(\mathcal{C})$; if $f\colon C \to C'$ is a morphism, then $\hom^A(f)\colon Mor(C', A) \to Mor(C, A)$ is defined by $\phi \mapsto \phi \circ f$ for all $\phi \in Mor(C', A)$.

Note that if $\mathcal{C} = \mathcal{A}b$, then $\hom_A$ and $\hom^A$ map into $\mathcal{A}b$.

Universals.

For a reference, see [J2, Chapter 1].

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F\colon \mathcal{C} \to \mathcal{D}$ a covariant functor. Let $B \in ob(\mathcal{D})$. A *universal* from B to the functor F is a pair (U, u) where $U \in ob(\mathcal{C})$ and $u \in Mor_{\mathcal{D}}(B, F(U))$ such that if $g \in Mor_{\mathcal{D}}(B, F(A))$, then there exists a unique $\tilde{g} \in Mor_{\mathcal{C}}(U, A)$ such that $F(\tilde{g})u = g$. U is called a *universal object* for B and u is the corresponding *universal map*.

Examples.

(1) Let $N \lhd G$ be groups. Take $\mathcal{C} = \mathcal{G}p$ and $\mathcal{D} =$ pairs of groups (H_1, H_2) with $H_2 \lhd H_1$ and a morphism $(H_1, H_2) \to (G_1, G_2)$ being a group homomorphism $H_1 \to G_1$ such that H_2 maps into G_2. Our functor F is defined by $F(G) = (G, \{e\})$ and $F(f) = f$ for any group homomorphism f. Now, let $B = (G, N)$ be our object in $\mathcal{D}$. The universal object is $U = G/N$ with universal map $u\colon (G, N) \to (G/N, \{e\})$. For any $H \in ob(\mathcal{G}p)$, if $g \in Mor_{\mathcal{D}}((G, N), (H, \{e\}))$ (i.e. $N \subseteq \ker g$), then there exists a unique group homomorphism $\tilde{g}\colon G/N \to H$ such that $(G, N) \to (G/N, \{e\}) \to (H, \{e\})$ is the mapping g.

(2) (See [S, Cor. 7.11.11, p. 297].) Let F be the inclusion of the category $\mathcal{F}ield$ of fields into the category $\mathcal{D}om$ of integral domains and ring monomorphisms (injections). For $D \in ob(\mathcal{D}om)$, the universal object is its field of fractions F.

Theorem (Uniqueness of universals). *If (U, u) and (U', u') are two universals from an object B to a functor F, then there exists a unique isomorphism $h\colon U \to U'$ such that $u' = F(h)u$.*

Definition (dual to universal). Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F\colon \mathcal{C} \to \mathcal{D}$ a covariant functor. Let $B \in ob(\mathcal{D})$. A *couniversal* from B to the functor F is a pair (U, u) where $U \in ob(\mathcal{C})$ and $u \in Mor_{\mathcal{D}}(F(U), B)$ such that if $g \in Mor_{\mathcal{D}}(F(A), B)$, then there exists a unique $\tilde{g} \in Mor_{\mathcal{C}}(A, U)$ such that $uF(\tilde{g}) = g$. U is called a *couniversal object* for B and u is the corresponding *couniversal map.*

Couniversal objects are also unique up to isomorphism with a similar proof. We now look at how these concepts give us many important concepts in algebra.

Coproducts.

Let $\{A_i\}_{i\in I}$ be a set of objects in a category $\mathcal{C}$ indexed by the set I. Let $\mathcal{D}$ be the class of all objects of the form $(C_i)_{i\in I}$, $C_i \in ob(\mathcal{C})$. Define the functor $F\colon \mathcal{C} \to \mathcal{D}$ by the diagonal embedding $F(C) = (C)_{i\in I}$. For a morphism $f \in Mor_{\mathcal{C}}(C_1, C_2)$, we define $F(f) = \{\, f\colon C_1 \to C_2 \text{ for each } i\,\}$. A universal object for $(A_i)_{i\in I}$ is called the *coproduct* of the objects A_i and is denoted by $\displaystyle\bigoplus_{i\in I} A_i$ or $\displaystyle\coprod_{i\in I} A_i$.

Thus $\bigoplus A_i$ is an object in $\mathcal{C}$ and comes with $u \in Mor_{\mathcal{D}}((A_i), F(U))$, that is u is a collection of morphisms $u_i\colon A_i \to \bigoplus A_i$, such that for all $A \in ob(\mathcal{C})$ with $\{\, g_i\colon A_i \to A\,\}$, there exists a unique morphism $\tilde{g}\colon \bigoplus A_i \to A$ such that $\tilde{g}u_i = g_i$ for all i.

Example: $\bigoplus_{i\in I} \mathbb{R} = \mathbb{R}^{(I)} = \{\, f\colon I \to \mathbb{R} \mid f \text{ has finite support }\}$. This is a real vector space and I corresponds to a basis.

Products.

Let $\{A_i\}_{i\in I}$ be a set of objects in a category $\mathcal{C}$ indexed by the set I. Let $\mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{D}$ be as before. A couniversal object for $(A_i)_{i\in I}$ is called the *product* of the objects A_i and is denoted by $\displaystyle\prod_{i\in I} A_i$.

The corresponding morphism $u \in Mor_{\mathcal{D}}(F(U), (A_i))$, is a collection of morphisms $u_i\colon \prod A_i \to A_i$ called *projection mappings*. Thus, for all $A \in ob(\mathcal{C})$ with morphisms $\{\, g_i\colon A \to A_i\,\}$, there exists a unique morphism $\tilde{g}\colon A \to \prod A_i$ such that $u_i\tilde{g} = g_i$ for all i.

Example: $\prod_{i\in I} \mathbb{R} = \mathbb{R}^I = \{\, f\colon I \to \mathbb{R}\}$. This is a real vector space and which is much larger than the coproduct. For example, if $I = \{0, 1, 2, \dots\}$, then $R^{(I)}$ is isomorphic to the vector space of polynomials $\mathbb{R}[x]$, while R^I is isomorphic to the vector space of formal power series $\mathbb{R}[[x]]$, and has an uncountable basis.

Example: Product in $\mathcal{S}ets$ is the usual Cartesian product. Coproduct in $\mathcal{S}ets$ is disjoint union of the sets S_i together with the canonical injections $\iota_j\colon S_j \to \dot{\bigcup} S_i$. Given functions

$\beta_j \colon S_j \to S$, we obtain a unique function γ from the disjoint union $\dot{\bigcup} S_i$ to S defined by $\gamma(s) = \beta_j(s)$ for $s \in S_j$ (i.e., $s = \iota_j(s)$).

We know that products exist in $\mathcal{G}p$: just take the set product and define the operation componentwise. The coproduct in $\mathcal{G}p$ is called the *free product* of groups.

Free products of groups [S, §6.3]**.**

Let H, K be groups and set $X = H \dot{\cup} K$. A *word* in X is a formal product $x_1 * x_2 * \cdots * x_k$, $x_i \in X$. The *empty word* has no elements in its product ($k = 0$).

Reduction of words: If two elements x_i, x_{i+1} are in the image of $\iota_H \colon H \to X$, then replace $x_i * x_{i+1}$ by $x_i x_{i+1}$. Similarly, if two elements are in the image of ι_K. Also eliminate e_H and e_K (so they reduce to the empty word). These are *elementary reductions*. We say two words w, w' are equivalent if there is a sequence of words $w = w_0, w_1, \ldots, w_n = w'$ such that for each i, either w_i reduces to w_{i+1} or w_{i+1} reduces to w_i. A word is called *reduced* if no x_i equals e_H or e_K and no adjacent pair x_i, x_{i+1} come from the same group.

Lemma 6.3.12. *Given any word in X, there is a unique reduced word to which it reduces by elementary reductions.*

Define $H * K$ to be the set of equivalence classes of words. By the lemma, this corresponds to the set of reduced words. The operation on this set is defined to be concatenation followed by reduction. This makes $H * K$ into a group with e equal to the empty word and $(x_1 * x_2 * \cdots * x_k)^{-1}$ equal to the class of $x_k^{-1} * x_{k-1}^{-1} * \cdots * x_1^{-1}$. Write i_H and i_K for the inclusions of H and K into $H * K$. Note that these are injective.

Theorem. *The group $H * K$ together with i_H and i_K is the coproduct of H and K in $\mathcal{G}p$.*

Example. $\mathbb{Z}_2 * \mathbb{Z}_2$: write the groups as $\{e, s\}$ and $\{e, t\}$. The free product has elements $e =$ empty word, $s, t, s*t, t*s, s*t*s, \ldots$ and is an infinite nonabelian group. This example suggests that the free product of nontrivial groups is never abelian. Thus the coproduct in the category $\mathcal{A}b$ must be quite different. In fact, for abelian groups A_i, the coproduct is $\oplus_{i \in I} A_i = \{\, (a_i)_{i \in I} \mid \text{all but finitely many } a_i \text{ equal zero} \,\}$.

Free Objects.

Let $\mathcal{C}$ be a *concrete category*, that is, one in which the objects C have "underlying sets" $\mathcal{U}(C)$ and the morphisms f correspond to set functions $\mathcal{U}(f)$. (Examples: $\mathcal{G}p$, $\mathcal{A}b$, $\mathcal{R}ings$.)

Definition. A *free object* on a set X in $\mathcal{C}$ is a universal element for the forgetful functor $F \colon \mathcal{C} \to \mathcal{S}ets$.

Example. Let X be a set and the category be vector spaces over some field. The free object is a vector space V with a function $\alpha\colon X \to V$ such that for any vector space W and any set function $\beta\colon X \to W$, there exists a unique linear transformation $\gamma\colon V \to W$ such that $\beta = \gamma\alpha$. But this just says that $\alpha(X)$ is a basis for V. Since every vector space has a basis, every vector space is free.

Free Groups.

For groups, the requirement that F_X be a free group on the set X says that there is a mapping $\alpha\colon X \to F_X$ and for any group G and any set function $\beta\colon X \to G$, there exists a unique group homomorphism $\gamma\colon F_X \to G$ such that $\beta = \gamma\alpha$.

Theorem. *For any set X, there exists a unique free group (F, α) on X. Furthermore α is injective and F is generated by the image of α.*

Given two sets X, Y and a map $f\colon X \to Y$, we have free groups F_X and F_Y and, from the property of being free, a unique homomorphism $F_X \to F_Y$ such that $X \to F_X \to F_Y$ is the same as $X \to Y \to F_Y$. Thus we may regard F as a functor from $\mathcal{S}ets$ to $\mathcal{G}p$ taking each set to a free group on that set. If $|X| = n$, we say that F_X is the free group on n generators.

Proposition. *Every group is a quotient group of a free group.*

Definition. A set X together with a subset $R \subseteq F_X$ is called a *presentation* for a group G if $G \cong F_X/\langle\langle R\rangle\rangle$, where $\langle\langle R\rangle\rangle$ is the normal subgroup of F_X generated by R. The elements of X are called *generators* and the elements of R are called *relators*.

Examples.
(1) $\langle x \mid x^n\rangle \cong \mathbb{Z}_n$.
(2) $\langle x, y \mid xyx^{-1}y^{-1}\rangle \cong \mathbb{Z} \times \mathbb{Z}$.
(3) $\langle x, y \mid \;\rangle$ is the free group on two generators.
(4) The free product of $\langle X_1 \mid R_1\rangle$ and $\langle X_2 \mid R_2\rangle$ is $\langle X_1 \dot\cup X_2 \mid R_1 \cup R_2\rangle$.

An excellent reference for the study of presentations is [MKS].

References

[J2] N. Jacobson, *Basic Algebra II*, W. H. Freeman and Co., 1980.

[MKS] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Wiley & Sons, 1966.

[S] M. Steinberger, *Algebra*, PWS Publishing Co., Boston, 1994.