CHARACTERIZATION OF FANS IN *-FIELDS

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Brocker's valuation-theoretic characterization of fans in a formally real field is extended to skew fields with involution. It is shown how each fan of a *-field arises as the pullback of a trivial fan in the residue skew field of a valuation.

1. Introduction and notation

In a formally real field, a fan is a certain special intersection of positive cones of orderings. Fans are very important to the study of quadratic forms over the field. The current status of the theory for commutative fields is well summarized in [9].

Our interest here will be in a *-field \((D, \ast)\); that is, a skew field \(D\) with an involution \(\ast\). The concept of *-ordering is due to Holland [8], though we shall use the definitions of [4,5]. For any subset \(A\) of \(D\), we shall write \(A^\ast\) for the set of nonzero elements in \(A\). We shall write \(S(D)\) for the set of symmetric elements \(\{d \in D \mid d = d^\ast\}\).

An extended *-ordering (called 'strong ordering' in [8]) is a subset \(P\) of \(D\) satisfying \(P \cup P \subseteq P; 1 \in P; dPd^\ast \subseteq P, \) for all \(d \in D^\ast; P \cup -P \supseteq S(D)^\ast; P \cap -P = \emptyset\) and \(P \cdot P \subseteq P\). A *-ordering is the set of symmetric elements in an extended *-ordering. These correspond best with orderings in the commutative case (with \(\ast = \text{identity}\)) since no one can be contained in another (see [3,4] for details). The importance of fans in this setting was demonstrated in [5, §6] where they are used to characterize the image of the Witt group of hermitian forms over \((D, \ast)\) inside the ring of continuous functions on the set of all *-orderings \(X_D\), where hermitian forms induce functions via their signatures. In fact, this set \(X_D\) forms a space of orderings in the sense of Marshall [10]. Fans are used in his abstract setting to prove the results necessary for [5] (see [11]). For this reason, it is important to understand their form as subsets of \(D\).

Given a *-ordering \(P\), we define \(P^e\) to be the maximal extended *-ordering containing \(P\). The set \(P^e\) can be explicitly defined in terms of the order valuation of \(P\); this is the valuation \(v_P\) associated to the ring of archimedean classes determined by \(P\), namely \(A(P) = \{d \in D \mid 0 \leq dd^\ast < n, \text{ for some } n \in \mathbb{Z}\}\). Holland [7,8] has shown
that this is a $\ast$-valuation ring (i.e. is closed under the involution $\ast$, or equivalently, $\nu_P(d) = \nu_P(d^\ast)$ for all $d \in D^\times$). The concept of valuation here is the usual one from [12], though we shall always add the additional condition that it respect the involution.

The role of the set of sums of squares in the commutative theory is played here by $\Sigma = \Sigma(D)$, the set of all sums of products of norms (i.e., elements of the form $dd^\ast$) and elements of the commutator group $[D^\times, I\Sigma(D)]$, where $I\Sigma(D)$ is the group of nonzero products of symmetric elements. We write $S(\Sigma)$ or $S(\Sigma(D))$ for $\Sigma(D) \cap S(D)$, which as in Artin–Schreier theory, turns out to be the intersection of all $\ast$-orderings [8].

Following [5, Definition 3.5], we define an extended preordering of $(D, \ast)$ to be a subset $T \subseteq D^\times$ satisfying $T + T \subseteq T$, $dT d^\ast \subseteq T (d \in D^\times)$, $\Sigma \subseteq T$, $-1 \notin T$ and $T \cdot T \subseteq T$. Again, a preordering is the set of symmetric elements in an extended preordering, and any preordering has a natural choice of extended preordering denoted $T^c$ such that $T^c \cap S(D) = T$ [5, Theorem 3.6]. The smallest extended preordering is $\Sigma$ and the smallest preordering is $S(\Sigma)$.

A preordering $T$ is called a fan if for each subgroup of $S(D)^\ast \cdot T^c (= I\Sigma(D) \cdot T^c$ by [5, Lemma 1.1]) of index two containing $T^c$ and excluding $-1$, the set of symmetric elements therein forms a preordering. For other equivalent characterizations of fans, see [5, Proposition 6.5] and Theorem 2.8 below.

The notion of compatibility between a $\ast$-ordering $P$ and a $\ast$-valuation $\nu$ is essentially the same as the commutative case [5, Theorem 4.2]. They are said to be compatible if $0 < a \leq b$ with respect to $P$ implies $\nu(a) \geq \nu(b)$ in the value group. A preordering $T$ is said to be compatible with a $\ast$-valuation if some $\ast$-ordering in $X_T$, the set of all $\ast$-orderings containing $T$, is compatible with $\nu$. In addition, $T$ is said to be fully compatible with $\nu$ if every $\ast$-ordering in $X_T$ is compatible with $\nu$.

Our main goal in this paper is to extend to $\ast$-fields the valuation theoretic characterization of fans due to Bröcker [1, §2] in which he shows that each fan arises as the pullback of a trivial fan on the residue field of some $\ast$-valuation. As for commutative fields, we shall call a fan trivial if it is a $\ast$-ordering or is the intersection of two $\ast$-orderings. The extension of Bröcker’s theorem is Theorem 2.13 below and an appropriate $\ast$-valuation is explicitly constructed in the proof of Theorem 2.12.

The paper concludes with an application of this theorem to $\ast$-fields in which the preordering $S(\Sigma)$ is a fan. Following the commutative terminology introduced by Brown [2], we call these superordered $\ast$-fields.

2. Fans and valuations

Let $T$ be a preordering of $(D, \ast)$. Let $X_T^\ast$ denote the set of $\ast$-orderings containing $T$ and compatible with a $\ast$-valuation $\nu$. In [4, Theorem 3.4] we have generalized the usual correspondence for commutative fields between orderings compatible with a valuation and orderings of the residue class field. In this paper we shall need to
generalize slightly further to \(*\)-orderings in $X^v$. We recall from \cite[Definition 2.11]{2} that a \(*\)-valuation $\nu$ is said to be real if $0 \in \Sigma(D_0)$ and the image of $[D^\times, S(D)^\times]$ is contained in $\Sigma(D_0)$. We note that the latter condition can be weakened somewhat.

**Proposition 2.1.** Let $\nu$ be a \(*\)-valuation for which $[D^\times, S(D)^\times]$ has image contained in some extended \(*\)-ordering $\mathcal{P}^e$ of the residue field $D_0$. Then $\Sigma(D) = \Sigma(D_0)$, $S(\Sigma(D)) = S(\Sigma(D_0))$ and $\nu$ is a real \(*\)-valuation compatible with $S(\Sigma)$.

**Proof.** First we note that $\Sigma(D) \subseteq \Sigma(D_0)$. For, if $d = \sum a_i c_i \in \Sigma(D) \cap U_0$, $a_i \in D^\times$, $c_i \in [D^\times, ITS(D)]$, then either all the terms lie in $A_0$, in which case we immediately obtain $d \in \Sigma(D_0)$, or we divide by the term of minimum value and reduce to obtain $0 \in \Sigma(D_0)$, a contradiction of the existence of a \(*\)-ordering on $D_0$. This implies that $\nu$ is real. Restricting our attention to symmetric elements yields $S(\Sigma(D)) = S(C(D_0))$. In particular, $-1 \in S(\Sigma(D))$ and it is easily seen that $S(X(D))$ is a preordering of $D$. Since $S(\Sigma(D_0))$ is also a preordering, we have $\nu$ compatible with $S(\Sigma(D))$ by \cite[Proposition 4.5]{1}.\

One consequence of this proposition is that every \(*\)-ordering of the residue field lifts to $D$, which is what one would expect from the commutative case. However, this is not true for orderings of skew fields without involution \cite{6}.

The classical theorem of Baer and Krull \cite[Theorem 3.18]{9} gives a one-to-one correspondence between orderings of a field compatible with a real valuation $\nu$ and pairs consisting of an ordering of the residue class field and a character of the value group modulo even elements. This was extended to \(*\)-fields in \cite[Theorem 3.4]{4}. We now need a more general version for $X^v$. Let $\Gamma$ be the value group of a real \(*\)-valuation $\nu$ and write $S(\Gamma)$ for the set $\nu(S(D)^\times)$, a subgroup of $\Gamma$ because $\nu$ is real \cite{4}. As in the classical situation, we must find some kind of a section $S(\Gamma) \rightarrow S(D)^\times$. The proof of our extension of \cite[Theorem 3.4]{4} is the same as the original proof, with the exception that we must replace the semisection of that argument by what we shall call a $T$-semisection. This is a mapping $s : S(\Gamma) \rightarrow S(D)^\times$ satisfying the properties of \cite[Theorem 2.7]{3} (in particular, $\nu(s(\gamma)) = \gamma$) and the new property that for $t \in T$, we have $s(\nu(t)) \in T$. We shall give an explicit construction of the mapping $s$.

First set $s(0) = 1$. For $\delta \in \Gamma$, choose $d \in D^\times$ such that $\nu(d) = \delta$ and set $s(2\delta) = dd^*$. Let $\{a_i\}$ be a set of representatives in $\nu(T)$ for a $\mathbb{Z}/2\mathbb{Z}$-basis of $\nu(T)/2\Gamma$. For each $i$, choose $s(a_i) \in T$ with $\nu(s(a_i)) = a_i$. Let $\{\beta_i\}$ be representatives for a $\mathbb{Z}/2\mathbb{Z}$-basis of $S(\Gamma)/\nu(T)$ and choose each $s(\beta_i) \in S(D)$ with $\nu(s(\beta_i)) = \beta_i$. Now any element of $S(\Gamma)$ has a unique expression of the form $\sum \beta_i + \sum a_i + 2\delta$ and we define

$$s(\sum \beta_i + \sum a_i + 2\delta) = d(s(\beta_1) \cdots s(\beta_n) s(a_1) \cdots s(a_m)) + s(\alpha_m) \cdots s(\beta_1))d^*,$$

where $s(2\delta) = dd^*$. With this definition of $s$, Theorem 2.7 of \cite{3} again holds and now, for $t \in T$, $s(\nu(t)) \in T$. The proof of \cite[Theorem 3.4]{4} now carries over to give a proof of the following:
Theorem 2.2. Let \( \nu \) be a real *-valuation on \((D, \ast)\) with \( T \)-semisection \( s : S(\Gamma) \to D^\times\).

1. Let \( P \in X_T^\nu \). Then \( \overline{P} = \{ a \in D^n_+ \mid a \in P, \nu(a) = 0 \} \) lies in \( X_T \) and \( \sigma(\gamma)s(\gamma) \in P \) for all \( \gamma \in s(\Gamma) \) defines a homomorphism \( \sigma : S(\Gamma)/v(T) \to \{ \pm 1 \} \).

2. Let \( \overline{P} \in X_T \) and \( \sigma \in \text{Hom}(S(\Gamma)/v(T), \{ \pm 1 \}) \). Then

\[
P = \{ a \in S(D) \mid \frac{as(\nu(a))^{-1}}{\sigma(\nu(a))} \in \overline{P} \}
\]
is in \( X_T^\nu \).

3. The processes of statements (1) and (2) give a bijective correspondence between the sets \( X_T^\nu \) and \( X_T \times \text{Hom}(S(\Gamma)/v(T), \{ \pm 1 \}) \). Furthermore, this bijection is a homeomorphism when the sets of *-orderings are given their usual topologies (cf. [4, \S 4]) and the character group \( \text{Hom}(S(\Gamma)/v(T), \{ \pm 1 \}) \) is given the character group topology. \( \square \)

The final statement on the topologies is almost immediate from the definitions. In the notation of [4, and writing \( G^* \) for the character group \( \text{Hom}(S(\Gamma)/v(T), \{ \pm 1 \}) \), the clopen set \( H(a) \subset X_T \), for \( a \in S(D)^\times \), corresponds to

\[
\{ \sigma \in G^* \mid \sigma(\nu(a)) = 1 \} \times \overline{H(\frac{as(\nu(a))^{-1} + s(\nu(a))^{-1}}{a})}
\]
\[
\cup \{ \sigma \in G^* \mid \sigma(\nu(a)) = -1 \} \times \overline{H(\frac{-as(\nu(a))^{-1} - s(\nu(a))^{-1}}{a})}.
\]

Lemma 2.3. Let \( d \in \text{IPS}(D) \). Then \( d^{-1}d^\ast \) lies in the commutator group \([\text{IIS}(D), S(D)^\times]\).

Proof. Since \( d \in \text{IPS}(D) \), it can be written in the form \( d = s_1s_2 \ldots s_n \), \( s_i \in S(D) \). We then have \( d^{-1}d^\ast - s_n^{-1} \ldots s_1^{-1}s_2 \ldots s_n - 1 \) modulo commutators of symmetric elements and \( \text{IPS}(D) \). \( \square \)

Lemma 2.4. Let \( \nu \) be a real *-valuation. Any unit in \( \text{IPS}(D) \) can be written as the product of a symmetric unit and an element of \( U_{\nu} \cap \text{IPS}(D) \cap \Sigma(D) \), where \( U_{\nu} \) denotes the units in the valuation ring of \( \nu \).

Proof. Let \( u \in U_{\nu} \cap \text{IPS}(D) \). We can write \( u = (u + u^\ast)(1 + u^{-1}u^\ast)^{-1} \), where \( u + u^\ast \) lies in \( S(D) \) and is a unit by [3, Lemma 2.6]. Also, \( u^{-1}u^\ast \in \Sigma(D) \) by Lemma 2.3, hence \( (1 + u^{-1}u^\ast)^{-1} \in \Sigma(D) \). Since \( (1 + u^{-1}u^\ast)^{-1} = (u + u^\ast)^{-1}u \), it also lies in \( \text{IPS}(D) \cap U_{\nu} \). \( \square \)

Theorem 2.5. Let \( T \) be a preordering of \((D, \ast)\) compatible with a *-valuation \( \nu \). Write \( \Gamma \) for the value group of \( \nu \) and \( U \) for the group of units. Then

1. \( |X_T| \leq |X_T^\nu| = |S(\Gamma)/v(T)| \cdot |X_T^\nu| \).

2. \( [\text{IIS}(D) : \text{IIS}(D) \cap \text{IPS}(D)] \geq |S(\Gamma)/v(T)| \cdot [\text{IIS}(D_{\nu}) : \text{IIS}(D_{\nu}) \cap \text{IPS}(D_{\nu})] \).

If \( \nu \) is fully compatible with \( T \), these inequalities are equalities. Conversely, if the left hand side of (2) is finite and either (1) or (2) is an equality, then the *-valuation \( \nu \) is fully compatible with \( T \).
Proof. Note first that \( v \) is real since it is compatible with a preordering. All statements regarding \( |X_T| \) follow from Theorem 2.2. For the remainder, we consider two exact sequences. The valuation, restricted to \( \mathcal{I}(D) \), induces a short exact sequence

\[
1 \rightarrow (U \cap \mathcal{I}(D))/(U \cap \mathcal{I}(D) \cap T^c) \rightarrow \mathcal{I}(D)/(\mathcal{I}(D) \cap T^c) \\
\rightarrow s(\Gamma)/v(T) \rightarrow 1.
\]

We obtain another short exact sequence induced by the projection to the residue (skew) field of \( \mathcal{I}(D) \cap U \); note that Lemma 2.4 assures us that elements of this group can be written as symmetric units modulo \( T^e \), so that we have

\[
1 \rightarrow U_0/(\mathcal{I}(D) \cap U \cap T^e) \rightarrow (\mathcal{I}(D) \cap U)/(\mathcal{I}(D) \cap U \cap T^e) \\
\rightarrow \mathcal{I}(D_0)/(\mathcal{I}(D_0) \cap T^e) \rightarrow 1,
\]

where \( U_0 = \mathcal{I}(D) \cap (1 + m_0) \cdot (\mathcal{I}(D) \cap U \cap T^e) \). Using these two sequences, we obtain

\[
[\mathcal{I}(D) : \mathcal{I}(D) \cap T^e] = |S(\Gamma)/v(T)| \cdot |(U \cap \mathcal{I}(D))/(U \cap \mathcal{I}(D) \cap T^e)| \\
= |S(\Gamma)/v(T)| \cdot |\mathcal{I}(D_0)/(\mathcal{I}(D_0) \cap T^e)| \\
\cdot |U_0/(\mathcal{I}(D) \cap U \cap T^e)| \\
\geq |S(\Gamma)/v(T)| \cdot |\mathcal{I}(D_0) : \mathcal{I}(D_0) \cap T^e|,
\]

which is (2). From [5, Proposition 4.5, Theorem 4.1 and Theorem 3.9], we know that \( v \) is fully compatible with \( T \in \mathcal{I}(D) \cap (1 + m_0) \subseteq \mathcal{I}(D) \cap T^c \Rightarrow |U_0/(\mathcal{I}(D) \cap U \cap T^e)| = 1 \). This implies equality in (2), and the converse implication holds if the left-hand side of (2) is finite.

Let \( T \) be a preordering of \((D, \ast)\). We now turn our attention to certain specific \( \ast \)-valuations known to be compatible with \( T \). For any \( \ast \)-ordering \( P \in X_T \), the valuation \( v_P \) defined in the introduction is compatible with \( P \) [8], and hence compatible with \( T \). We shall write \( A_T \) for the \( \ast \)-valuation ring

\[
\prod A(P) \quad (P \in X_T),
\]

where the product is the operation inside \( D \). We write \( v_T \) for the corresponding \( \ast \)-valuation. For each \( P \in X_T \), we have \( 1 + (m_{v_T} \cap S(D)) \subseteq 1 + (m_{v_T} \cap S(D)) \subseteq P \), and hence contained in the intersection \( \bigcap_{P \in X_T} P \), which equals \( T \) by [5, Corollary 3.14]. This shows that \( v_T \) is fully compatible with \( T \) [5, Proposition 4.5].

Proposition 2.6. (1) The valuation \( v_T \) is the finest \( \ast \)-valuation of \((D, \ast)\) fully compatible with \( T \).

(2) For any set \( B \subseteq D \), write \( \overline{B} \) for the image in \( D_{v_T} \) of \( B \cap U_{v_T} \). For \( P \in X_T \), we have \( A(P) = \overline{A(P)} \).

(3) \( A_T = D \), or equivalently, the residue field has no nontrivial \( \ast \)-valuations fully compatible with \( T \).
Proof. (1) Assume that $A$ is a $*$-valuation ring properly contained in $A_T$. By definition of $A_T$, there exists a $*$-ordering $P \in X_T$ such that $A(P) \not\subseteq A$. We claim that $A$ is not compatible with $P$. Choose an element $x \in A(P)$, $x \notin A$. By definition of $A(P)$, there exists a positive integer $n$ such that $0 < xx^* < n$ with respect to $P$. Since $A$ is a $*$-valuation ring, we have $xx^* \notin A$, and hence $A$ is not compatible with $P$.

(2) By definition $A(\overline{P}) = \{ d \in D \mid 0 \leq dd^* < n, \text{for some } n \in \mathbb{Z} \}$. Given $a \in A(P)$, lift to $a \in D$. Then, for some positive integer $n$, $0 < aa^* < n$ with respect to $P$, so $a \in A(\overline{P})$ and $\overline{a} \in A(\overline{P})$. Conversely, if $\overline{a} \in A(\overline{P})$, then for any lifting $a \in A(P)$, we obtain an inequality $0 \leq aa^* < n$ which reduces to show $\overline{a} \in A(\overline{P})$.

(3) By definition, $A_T = \prod \{ A(P) \mid P \in X_T \}$. By Theorem 2.2, the mapping $P \rightarrow P$ from $X_T$ to $X_T$ is surjective, so (2) gives $A_T = \prod \{ A(P) \mid P \in X_T \} = \prod \{ A(P) \mid P \in X_T \} = A_T = D$.

We recall from [8] that, given a preordering $T$ and a nonzero symmetric element $a \notin T$, one obtains a larger preordering as

$$T[a] = \{ t_1 + t_2 a \mid t_1 \in T^* \cup \{0\} \cap S(D)^x \}.$$ 

Proposition 2.7. Let $T$ be a preordering with $|IS(D)/(IS(D) \cap T^x)|$ greater than 4. Let $P \neq Q$ be elements of $X_T$. Let $a \in P \cap Q$, with $a \notin T$. Let $v : D^x \rightarrow T$ be the order valuation of $P$. If $v(T) = S(T)$, then there exists $b \in T \setminus [a]$ such that $b$ lies in $P$ or in $Q$, but not both.

Proof. Write $U_T$ for the units in $A(P)$. Since $v(a) \in S(T) = v(T)$, we can write $a = tu$ for some $t \in T, u \in U_T$. Thus $u = t^{-1} a$ lies in $P^c \cap Q^c$ but not in $T^c$ and so $u_0 = u + u^*$ lies in $P \cap Q$, but not in $T$. It is easy to check that $T[-a] = T[-u_0]$, so we may assume without loss of generality that $a \in U_T$. First assume that $a \in U_Q$. We may replace $a$ by $a^{-1}$ if necessary to assume that $a \in A(Q)$. Choose $n \in \mathbb{Z}^+$ with $n > a$ with respect to $P$ and set $b = n - a \in T[-a]$. Then $b \in P \cap Q$ as desired. Next assume that $a \notin U_Q$. Write $\kappa_P$ and $\kappa_Q$ for the places corresponding to $v_P$ and $v_Q$, respectively. Then $r_1 = \pi_P(a)$ and $r_2 = \pi_Q(a)$ may be considered to be positive real numbers. (They are symmetric elements in the residue fields of order valuations which can be embedded in the real quaternions [7,].) We distinguish two cases:

Case 1. $r_1 \neq r_2$. We may assume $0 < r_1 < r_2$. Choose a rational number $q$ such that $r_1 < q < r_2$ and set $b = q - a \in T[-a]$. Then we have $\pi_Q(b) < \pi_P(b) < \infty$, so that $b \in P \cap Q$.

Case 2. $r_1 = r_2$. Let $c \in P, c \notin Q$. As with $a$, we may assume without loss of generality that $c \in U_P$. Also, replacing $c$ by $c^{-1}$ if necessary, we may assume $c \in A(Q)$. Set $r_1 = \pi_P(c) > 0$ in $\mathbb{R}$ and $r_2 = \pi_Q(c) \leq 0$ in $\mathbb{R}$. Choose an integer $n > -r_2$ so that $0 < 1 + r_2/n < 1 + r_1/n$. Now replace $a$ by $a(1 + c/n)^2 + (1 + c/n)^2 a$, again an element of $P \cap Q$ but not in $T$ because $(1 + c/n)^2 \in T$. Then we have $\pi_P(a) \neq \pi_Q(a)$ so Case 1 applies.

We are now prepared to turn our attention to the special preorderings known as
fans. We begin with a very useful characterization of fans in terms of their additive behavior.

**Theorem 2.8.** Let $T$ be a preordering of $(D, \ast)$. The following statements are equivalent:

1. $T$ is a fan.
2. For all $b \in S(D)^\times$, if $b \notin T$, then $1 + b \in T \cup bT$.
3. For all $b \in S(D)^\times \cdot T^e$, if $b \in -T^e$, then $(T^e + bT^e) \cap S(D)^\times \cdot T^e \subseteq T^e \cup bT^e$.

**Proof.** (1) $\Rightarrow$ (2). Assume that $T$ is a fan; i.e., $X_T = \{ P \cap S(D)^\times \mid P \text{ is a subgroup of } S(D)^\times \cdot T^e \text{ of index } 2, T^e \subseteq P, -1 \notin P \}$. Let $b \in S(D)^\times$, with $b \notin T$ and consider the set $M = T^e \cup bT^e$. It is a multiplicative subgroup of $S(D)^\times \cdot T^e$ of index 2, containing $M$ and excluding $-1$. Now $M = \bigcap P$, where $P$ ranges over all subgroups of $S(D)^\times \cdot T^e$ of index 2 containing $M$ and excluding $-1$. By hypothesis, the symmetric elements in each such $P$ form a $\ast$-ordering, so $M \cap S(D)$ is a preordering. In particular, it is closed under addition, contains 1 and $b$ and thus contains $1 + b$. Since $b$ is symmetric, $1 + b$ lies in $T$ if it lies in $T^e$ and if it lies in $bT^e$, then $1 + b = b(1 + b^{-1}) \in bT$.

(2) $\Rightarrow$ (3). Let $b \in S(D)^\times \cdot T^e$, $t \in T^e$ such that $1 + bt \in S(D)^\times \cdot T^e$. Then $1 + bt$ has sign with respect to each $P^e$ for $P \in X_T$. By (2), we know that $1 + (bt + t^*b^*)/2 \in T \cup (bt + t^*b^*)T^e \subseteq T \cup bT^e$; the last containment follows from the fact that, writing $b = st_0$, $s \in S(D)^\times$, $t_0 \in T^e$, we have $bt + t^*b^* = b(t + t_0^{-1}[s^{-1}, t^*]t^*t_0^{-1}[t^{-1*}, s^{-1}]) \in bT^e$. Now if $(1 + bt)/2 + (1 + bt)^*/2 = 1 + (bt + t^*b^*)/2$ lies in $I$, then $1 + bt \in P$ for all $P \in X_T$, hence $1 + bt \in T^e$. And if it lies in $bT^e$, then $b^{-1}(1 + bt) \in P$ for all $P \in X_T$, hence $b^{-1}(1 + bt) \in T^e$ and $1 + bt \in bT^e$. Finally, for $t_1, t_2 \in T^e$ with $t_1 + t_2 \in S(D)^\times \cdot T^e$, we obtain $t_1 + bt_2 = (1 + bt_2t_1^-)t_1 \in T^e \cup bT^e$, as desired.

(3) $\Rightarrow$ (1). Let $P$ be a subgroup of index 2 in $S(D)^\times \cdot T^e$ containing $T^e$ and excluding $-1$. We must show that $P \cap S(D)^\times$ is a $\ast$-ordering. We shall show that it is closed under addition; all other properties of a $\ast$-ordering follow easily (cf. [5, Proposition 6.6]). Let $a, b \in P \cap S(D)^\times$. By (3), we have $(T^e + a^{-1}bT^e) \cap S(D)^\times \cdot T^e \subseteq T^e \cup a^{-1}bT^e$. Multiplication by $a$ yields $(aT^e + bT^e) \cap S(D)^\times \cdot T^e \subseteq aT^e \cup bT^e$, and thus $a + b$ lies in $(aT^e \cup bT^e) \cap S(D) \subseteq P$. □

Next we begin to look at how fans interact with valuations.

**Proposition 2.9.** Let $v$ be a $\ast$-valuation and let $T$ be a preordering of $(D, \ast)$.

1. If $v$ is compatible with $T$ and $T$ is a fan, then $T$ is a fan.
2. If $v$ is fully compatible with $T$, then $T$ is a fan iff $\bar{T}$ is a fan.

**Proof.** (1) Let $b \in U_v$ such that $b \in S(D_v)^\times$, $b \notin -\bar{T}$. We may assume $b \in S(D)^\times$ and, of course, $b \notin -T$. Since $T$ is a fan, the element $1 + b$ lies in $T$ or $bT$, and therefore its image lies in $T$ or $bT$. Thus $T$ is a fan by Theorem 2.8.

(2) Assume $1 + (m_v \cap S(D)) \subseteq T$ and $\bar{T}$ is a fan. Let $b \in S(D)^\times$ with $b \notin -T$. If
b∈m_0, then 1 + b ∈ T by compatibility. If b ∉ A_0, then 1 + b^{-1} ∈ 1 + (m_0 ∩ S(D)) ⊆ T, and hence 1 + b = b(1 + b^{-1}) ∈ bT. Finally, if b ∈ U_0, then 1 + b ∈ T ∪ bT since T is a fan. Using compatibility once more, we obtain 1 + b ∈ T ∪ bT.

It follows from [5, Proposition 6.5] that a fan T is trivial iff |S(D) \times T^c / T^c| ≤ 4; we shall call T nontrivial if |S(D) \times T^c / T^c| > 4. We shall see the importance of this distinction later in our main characterization theorem.

**Corollary 2.10.** Let T be a nontrivial fan and let P ∈ X_T with order valuation v_P : D^X → T. Then v_P(T) ≠ S(Γ). In particular, v_P is a nontrivial valuation and thus P is nonarchimedean.

**Proof.** Assume that v_P(T) = S(Γ) and choose an element a ∈ P ∩ Q, a ∉ T, where P, Q ∈ X_T. By Theorem 2.8 we have T[-a] ⊆ T^c ∪ -aT^c. But no element of T^c or -aT^c can separate the *-orderings P and Q, contradicting Proposition 2.7.

**Proposition 2.11.** Let T be a fan and let v be a *-valuation for which the subset S(Γ) of Γ is a subgroup. If v(T) contains no nontrivial convex subgroup of S(Γ), then v is fully compatible with T.

**Proof.** By [5, Proposition 4.5] it will suffice to show that t + (m_0 ∩ S(D)) ⊆ T for any t ∈ T ∩ U_0. Let t ∈ T ∩ U_0 and m ∈ m_0 ∩ S(D).

Case 1. v(m) ∉ v(T). In this case m ∉ T so Theorem 2.8 implies that t + m ∈ T^c ∪ mT^c. Since t + m is a unit and v(m) ∉ v(T), t + m cannot lie in mT^c and therefore must lie in T.

Case 2. v(m) ∈ v(T). Since v(T) contains no nontrivial convex subgroup of S(Γ), we can apply [9, Lemma 12.14] to obtain the existence of an element γ ∈ S(Γ), γ ∉ v(T) such that 0 < γ < v(m). Choose any d ∈ S(D) with v(d) = γ. Then, for t ∈ T ∩ U_0, we have t + m = (t + d) + (m - d) with t + d ∈ U_0 (since v(t) = 0, v(d) = γ > 0) and t + d ∈ T by Case 1 since v(d) ∉ v(T). Also, v(d) = γ < v(m) implies that v(m - d) = v(d) ∈ v(T), so m - d ∈ m_0 and we can again apply Case 1 to show that t + m lies in T.

**Theorem 2.12.** Let T be a nontrivial fan. Then A_T ⊆ D. In particular, there exists a nontrivial valuation fully compatible with T.

**Proof.** Let P be any *-ordering in X_T. By Corollary 2.10 we have v_P(T) ≠ S(Γ_P), where Γ_P denotes the value group of v_P. The convex subgroups of the ordered group S(Γ_P) lying inside v_P(T) form a chain under inclusion. Let A be the union of the chain, the largest convex subgroup of S(Γ_P) contained in v_P(T). Let A_0 = {γ ∈ Γ_P | γ ≤ δ, for some δ ∈ Δ}, a convex subgroup of Γ_P with ∆_0 ∩ S(Γ_P) = Δ. Next we coarsen v_P to a *-valuation v_0 : D → Γ_0 = Γ_P / ∆_0. The valuation v_0 is fully compatible with T by Proposition 2.11.
Characterization of fans in \(*\)-fields

\[
S(\mathcal{F}_0)/v_0(T) \equiv (S(\mathcal{F}_p)\Delta_0/\Delta_0)/(v_p(T)\Delta_0/\Delta_0)
\]
\[
\equiv (S(\mathcal{F}_p)/\Delta)/(v_p(T)/\Delta)
\]
\[
\equiv S(\mathcal{F}_p)/v_p(T)
\]

so that \([S(\mathcal{F}_0):v_0(T)] = [S(\mathcal{F}_p):v_p(T)] > 1\), and thus \(v_0\) is nontrivial. The valuation \(v_T\) is also nontrivial by Proposition 2.6 with \(A_T\) contained in the valuation ring of \(v_0\). □

Finally we come to our main theorem extending the work of Bröcker [1].

**Theorem 2.13.** Let \(T\) be a preordering of \((D, \ast)\) and set \(v = v_T\). The following statements are equivalent:

1. \(T\) is a fan.
2. \([\text{IS}(D_v): \mathcal{T}_v \cap \text{IS}(D_v)] \leq 4\) (i.e. the induced fan \(\mathcal{T}\) on \(D_v\) is trivial).
3. There exists a \(*\)-valuation \(v_0\) fully compatible with \(T\) such that \(T\) pushes down to a trivial fan on \(D_{v_0}\).
4. \([\text{IS}(D)/v(T)] \geq [\text{IS}(D): \mathcal{T}_v \cap \text{IS}(D)]/4\).

**Proof.** (1) \(\Rightarrow\) (2). By Proposition 2.9(1), we know that \(\mathcal{T}\) is a fan in \(D_v\). If \([\text{IS}(D_v): \mathcal{T}_v \cap \text{IS}(D_v)] \geq 8\), then \(\mathcal{T}\) is nontrivial and Theorem 2.12 implies there exists a nontrivial valuation on \(D_v\) fully compatible with \(\mathcal{T}\). But this is impossible since \(A_T = D_v\) by Proposition 2.6(3). Therefore \([\text{IS}(D_v): \mathcal{T}_v \cap \text{IS}(D_v)] \leq 4\).

(2) \(\Rightarrow\) (1). By Proposition 2.9(2), we know that \(T\) is a fan.

(2) \(\Rightarrow\) (4). Proposition 2.6(1) tells us that \(v\) is fully compatible with \(T\). Applying Theorem 2.5, we obtain

\[
[S(\mathcal{F}_v)/v(T)] = [\text{IS}(D): \mathcal{T}_v \cap \text{IS}(D)]/[\text{IS}(D_v): \mathcal{T}_v \cap \text{IS}(D_v)]
\]
\[
\geq [\text{IS}(D): \mathcal{T}_v \cap \text{IS}(D)]/4.
\]

(4) \(\Rightarrow\) (2). Again using Theorem 2.5,

\[
[S(\mathcal{F}_v)/v(T)] \geq [\text{IS}(D): \mathcal{T}_v \cap \text{IS}(D)]/4
\]
\[
= [S(\mathcal{F}_v)/v(T)] \cdot [\text{IS}(D_v): \mathcal{T}_v \cap \text{IS}(D_v)]/4,
\]
and we obtain \([\text{IS}(D_v): \mathcal{T}_v \cap \text{IS}(D_v)] \leq 4\). □

3. **Superordered \(*\)-fields**

In [2], Brown has defined a superordered field as a formally real field \(F\) which has as many orderings as possible; i.e. every subgroup of \(F^\times\) of index 2 containing all nonzero sums of squares and not containing \(-1\) is an ordering. We introduce the corresponding concept for \(*\)-fields.
Definition 3.1. The *-field \((D, \ast)\) is superordered if the set of symmetric elements in \(P\) is a *-ordering for every subgroup \(P\) of \(S(D)^\ast \cdot \Sigma(D)\) of index 2 containing \(\Sigma(D)\) and excluding \(-1\).

From [5, Proposition 6.5], we immediately obtain the following:

**Proposition 3.2.** The following statements are equivalent for the *-field \(D\):

1. \((D, \ast)\) is superordered.
2. \(S(\Sigma(D))\) is a fan.
3. The reduced Witt ring \(W_{\text{red}}(D, \ast)\) is isomorphic to an integral group ring. \(\Box\)

In (3), the group ring is in fact \(\mathbb{Z}[S(D)^\ast \cdot \Sigma/\pm \Sigma]\). One of Brown's main results is a valuation-theoretic characterization of superordered fields. The generalization of his theorem to our context is a consequence of Theorem 2.13.

**Proposition 3.3.** The *-field \(D\) is superordered iff there exists a *-valuation compatible with all *-orderings of \((D, \ast)\) such that \(D_v\) has at most two *-orderings.

**Proof.** This is just the equivalence of (1) and (3) of Theorem 2.13 for the pre-ordering \(S(\Sigma)\). \(\Box\)

**Corollary 3.4.** Let \(v\) be a *-valuation fully compatible with \(S(\Sigma)\). Then \(D\) is superordered iff \(D_v\) is superordered.

**Proof.** This follows immediately from Proposition 2.1, Proposition 2.9(2) and Proposition 3.2. \(\Box\)

**References**