COMPOSITION THEOREMS, MULTIPLIER SEQUENCES AND COMPLEX ZERO DECREASING SEQUENCES

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Abstract. An important chapter in the theory of distribution of zeros of polynomials and transcendental entire functions pertains to the study of linear operators acting on entire functions. This article surveys some recent developments (as well as some classical results) involving some specific classes of linear operators called multiplier sequences and complex zero decreasing sequences. This expository article consists of four parts: Open problems and background information, Composition theorems (Section 2), Multiplier sequences and the Laguerre-Pólya class (Section 3) and Complex zero decreasing sequences (Section 4). A number of open problems and questions are also included.

1. Introduction: Open problems and background information

In order to motivate and adumbrate the results to be considered in the sequel, we begin here with a brief discussion of some basic (albeit fundamental) questions and open problems. Let π_n denote the vector space (over $\mathbb{R}$ or $\mathbb{C}$) of all polynomials of degree at most n. For $S \subseteq \mathbb{C}$ (where S is an appropriate set of interest), let $\pi_n(S)$ denote the class of all polynomials of degree at most n, all of whose zeros lie in S. (The problems cited in the sequel are all open problems.)

Problem 1.1 Characterize all linear transformations (operators)

$$T : \pi_n(S) \to \pi_n(S), \tag{1.1}$$

where, for the sake of simplicity, we will assume that $\deg T[p] \leq \deg p$.

Remarks. We hasten to remark that Problem 1.1 is open for all but trivial choices of S (and perhaps, for this reason, it has never been stated in the literature, as far as the authors know). In fact, this problem is open in such important special cases when (i) $S = \mathbb{R}$, (ii) S is a half-plane, (iii) S is a sector centered at the origin, (iv) S is a strip, say, $\{z \mid |\operatorname{Im} z| \leq r\}$, or to cite a non-convex, but important, example (v) S is a double sector centered at the origin and symmetric about the real axis. New results about classes of polynomials are almost always of interest; but when such new results also extend, say, to transcendental entire functions, they tend to be significant. For example, when S is the open upper half-plane, the Hermite-Biehler theorem [64, p. 13] characterizes the polynomials all whose zeros lie in S. Moreover, this theorem extends to certain transcendental entire functions Levin [60, Chapter VII]. If S is the left half-plane, then results relating to Problem 1.1 would be important in several areas of applied mathematics (see for example, Marden's discussion of dynamic stability [62, Chapter IX]). In this case, the known characterization of the *Hurwitz polynomials* (that is, real polynomials whose zeros all lie in the left half-plane [62, p. 167]) is undoubtedly relevant. (See also the work of Garloff and Wagner [37] concerning the Hadamard products of stable polynomials.)

It is interesting to note from an historical perspective, that finding just one new T satisfying (1.1) can be significant. For example, if S is a convex region in $\mathbb{C}$ and $T = D$, where $D = \frac{d}{dz}$, then by the classical Gauss-Lucas theorem T satisfies (1.1) (cf. [62, p. 22]). In the sequel, as we consider some special cases of Problem 1.1, we will encounter some other notable linear transformations which satisfy (1.1).

Problem 1.2 Characterize all linear transformations (operators) $T : \pi_n \to \pi_n$ such that

$$Z_c(T[p(x)]) \leq Z_c(p(x)), \tag{1.2}$$

where $p(x)$ and $T[p(x)]$ are *real polynomials* (that is, the Taylor coefficients of $p(x)$ are all real) and $Z_c(P(x))$ denotes the number of nonreal zeros of $p(x)$, counting multiplicities.

If $T = D = \frac{d}{dx}$, then (1.2) is a consequence of Rolle's theorem. If $q(x)$ is a real polynomial with only real zeros and $T = q(D)$, then (1.2) follows from the classical Hermite-Poulain Theorem [64, p. 4]. There are many other linear transformations T which satisfy inequality (1.2). Indeed, set

$T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in \mathbb{R}$, and for an arbitrary real polynomial $p(x) = \sum_0^n a_k x^k$, define

$$T[p(x)] := \sum_{k=0}^{n} \gamma_k a_k x^k. \tag{1.3}$$

If $Q(x)$ is a real polynomial with only real negative zeros and if $T = \{Q(k)\}_{k=0}^{\infty}$, then by a theorem of Laguerre (cf. Theorem 4.1 below)

$$Z_c\left(\sum_{k=0}^{n} Q(k) a_k x^k\right) \leq Z_c(p(x)),$$

where $p(x) = \sum_0^n a_k x^k$ is an arbitrary real polynomial.

Of course, differentiation is a linear transformation satisfying (1.2) and more: The polynomial $T[p(x)]$ has zeros between the real zeros of $p(x)$. In [21] the following problem is raised.

Problem 1.2a Characterize all linear transformations $T : \pi_n \to \pi_n$ such that $T[p(x)]$ has at least one real zero between any two real zeros of $p(x)$.

This problem is solved in [21, Corollary 2.4] for linear transformations defined as in (1.3). They are precisely those for which $\{\gamma_k\}_{k=0}^{\infty}$ is a nonconstant arithmetic sequence all of whose terms have the same sign.

Problem 1.3 Characterize all linear transformations (operators) $T : \pi_n \to \pi_n$ such that

if $p(x)$ has only real zeros, then $T[p(x)]$ also has only real zeros. (1.4)

Recently, a number of significant investigations related to the above problems have been carried out by Iserles and Saff [49], Iserles and Nørsett [47] and Iserles, Nørsett and Saff [48]. In particular, in [47] and [48] the authors study transformations that map polynomials with zeros in a certain interval into polynomials with zeros in another interval. In [18], Carnicer, Peña and Pinkus characterize a class of linear operators T (which correspond to unit lower triangular matrices) for which the degree of the polynomials p and $T[p]$ are the same and $Z_c(T[p]) \leq Z_c(p)$.

A noteworthy special case of Problem 1.3 arises when the action of the linear transformation T on the monomials is given by $T[x^n] = \gamma_n x^n$, for some $\gamma_n \in \mathbb{R}$, $n = 0, 1, 2, \ldots$. The transformations $T = \{\gamma_k\}_{k=0}^{\infty}$ which satisfy (1.4) are called *multiplier sequences* (cf. [73] or [72, pp. 100–124]). The precise definition is as follows.

Definition 1.4 A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence* if, whenever the real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has *only* real zeros, the polynomial $T[p(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$ also has *only* real zeros.

In 1914 Pólya and Schur [73] completely characterized multiplier sequences. Their seminal work was a fountainhead of numerous later investigations. Applications to fields other than $\mathbb{R}$ can be found in [19]. Among the subsequent developments, we single out the notion of a totally positive matrix and its variation diminishing property, which in conjunction with the work of Pólya and Schur, led to the study of the analytical and variation diminishing properties of the convolution transform by Schoenberg [77] and Karlin [50]. A by-product of this research led to conditions for interpolation by spline functions due to Schoenberg and Whitney [80]. (In regard to generating functions of totally positive sequences see, for example, [1], [2], [50]. Concerning the generating functions of Pólya frequency sequences of finite order, see the recent paper of Alzugaray [3]).

In light of Problem 1.2, it is natural to consider those multiplier sequences which satisfy inequality (1.2). These sequences are called *complex zero decreasing sequences* and are defined as follows.

Definition 1.5 ([24]) A sequence $\{\gamma_k\}_{k=0}^{\infty}$ is said to be a *complex zero decreasing sequence*, or CZDS for brevity, if

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k x^k\right), \tag{1.5}$$

for any real polynomial $\sum_{k=0}^{n} a_k x^k$. (The acronym CZDS will also be used in the plural.)

As a special case of Problem 1.2 we mention the following open problem.

Problem 1.6 Characterize all complex zero decreasing sequences.

The aim of this brief survey is to provide a bird's-eye view of some of the classical results as well as recent developments related to the aforementioned open problems. Since the so-called composition theorems ([62, Chapter IV], [64, Kapitel II]) play a pivotal role in the algebraic characterization of multiplier sequences, in Section 2 we examine some sample results which lead to the composition theorems. While a detailed discussion of the composition theorems is beyond the scope of this article, in Section 2 we include a proof of de Bruijn's generalization of the Malo-Schur-Szegö Composition Theorem. In Section 3 we state the Pólya and Schur algebraic and transcendental characterization of multiplier sequences [73]. The latter characterization involves a special class of entire functions known as the Laguerre-Pólya class. We exploit this connection and use it as a conduit in our formulation of a number of recently established properties of multiplier sequences. In Section 4 we highlight some selected results pertaining to the ongoing investigations of properties of CZDS and we list several open problems. Finally, we caution the reader that the selected bibliography is not intended to be comprehensive.

2. Composition theorems

A key step in the characterization of multiplier sequences rests on the composition theorems. In this section our aim is to succinctly outline some of the precursory ideas which lead to the Malo-Schur-Szegö Composition Theorem. Before stating this theorem, we briefly describe Laguerre's Separation Theorem and Grace's Apolarity Theorem, two results which are frequently invoked in the proofs of composition theorems for polynomials. (We remark parenthetically that there are other approaches to some of these theorems. Indeed, Schur's original proof ([81] or [60, p. 336]) was based on properties of Sturm sequences. However, Sturm sequences are inapplicable for the determination of the nonreal zeros of a polynomial and thus this approach does not seem to lend itself to generalizations.) Given the extensive literature dealing with composition theorems for polynomials (also called Hadamard products of polynomials), our treatment is of necessity perfunctory and is limited to our goal of providing a modicum of insight into the foundation of the theory of multiplier sequences. (For additional citations we refer to Borwein and Erdélyi [14], Marden [62] and Obreschkoff [64] and the references contained therein.)

In order to motivate Laguerre's Separation Theorem, we associate with each polynomial $f(z)$ a "generalized" derivative called the *polar derivative (with respect to* ζ*)*, $f_\zeta(z)$, defined by

$$f_\zeta(z) := nf(z) + (\zeta - z)f'(z), \qquad \text{where} \;\; \zeta \in \mathbb{C}. \tag{2.1}$$

Note that if $\deg f(z) = n$, then $f_\zeta(z)$ is a polynomial of degree $n-1$. When $\zeta = \infty$, then we define f_∞ to be the ordinary derivative. Now by the classical Gauss–Lucas Theorem [62, §6], any circle which contains in its interior all the zeros of a polynomial $f(z)$, also contains all the zeros of $f'(z)$. What is the corresponding result for polar derivatives? By considering circular regions (i.e., closed disks, or the closure of the exterior of such disks or closed half-planes), which are "invariant" under Möbius transformations, Laguerre obtained the following invariant form the Gauss-Lucas Theorem ([14, p. 20], [62, §13], [64, §4]).

Theorem 2.1 (Laguerre's Separation Theorem) *Let* $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \in \mathbb{C}$, *be a polynomial of degree* $n \geq 2$.

1. *Suppose that all the zeros of* f *lie in a circular region* D. *For* $\zeta \notin D$, *all of the zeros of the polar derivative* $f_\zeta(z) := nf(z) + (\zeta - z)f'(z)$ *lie in* D.

2. *Let α be any complex number such that $f(\alpha)f'(\alpha) \neq 0$. Then any circle, C, passing through the points α and $\alpha - \frac{nf(\alpha)}{f'(\alpha)}$ either passes through all the zeros of f or separates the zeros of f (in the sense that there is at least one zero of f in the interior of C and at least one zero in the exterior of C).*

Suppose that (for fixed ζ) $f_\zeta(\alpha) = 0$. Then, solving (2.1) for ζ in terms of α, we obtain (assuming that $f(\alpha)f'(\alpha) \neq 0$)

$$\zeta = \alpha - \frac{nf(\alpha)}{f'(\alpha)},$$

which appears as the "mysterious" point in Laguerre's Separation Theorem. Marden [62, p. 50] gives two proofs using spherical force fields and properties of the centroid of a system of masses. For a simple, purely analytical proof we refer to A. Aziz [5]. A masterly presentation of Laguerre's theorem, its invariance under Möbius transformations, (and some of its more recent applications) in terms of the notion of a generalized center of mass is given by E. Grosswald [39]. (See also Pólya and Szegö [74, Vol. II, Problems 101-120].)

In order to state Grace's Apolarity Theorem ([62, p. 61], [64, p. 23], [14, p. 23], [38]) it will be convenient to adopt the following definition.

Definition 2.2 Two polynomials

$$A(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \quad \text{and} \quad B(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k,$$

where $a_n b_n \neq 0$, are said to be *apolar* if their coefficients satisfy the relation

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k b_{n-k} = 0.$$

Theorem 2.3 (Grace's Apolarity Theorem) *Let $A(z)$ and $B(z)$ be apolar polynomials. If $A(z)$ has all its zeros in a circular region D, then $B(z)$ has at least one zero in D.*

Grace's Apolarity Theorem can be derived by repeated applications of Laguerre's Separation Theorem [62, p. 61]. This fundamental result relating the relative location of the zeros of two apolar polynomials, while remarkable for its lack of intuitive content, has far-reaching consequences. One such consequence is the following composition theorem.

Theorem 2.4 (The Malo-Schur-Szegö Theorem [62, §16], [64, §7]) *Let*

$$A(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \quad \text{and} \quad B(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k \tag{2.2}$$

and set

$$C(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k. \tag{2.3}$$

1. (Szegö, [85]) *If all the zeros of $A(z)$ lie in a circular region K, and if $\beta_1, \beta_2, \ldots, \beta_n$ are the zeros of $B(z)$, then every zero of $C(z)$ is of the form $\zeta = -w\,\beta_j$, for some j, $1 \le j \le n$, and some $w \in K$.*
2. (Schur, [81]) *If all the zeros of $A(z)$ lie in a convex region K containing the origin and if the zeros of $B(z)$ lie in the interval $(-1, 0)$, then the zeros of $C(z)$ also lie in K.*
3. *If the zeros of $A(z)$ lie in the interval $(-a, a)$ and if the zeros of $B(z)$ lie in the interval $(-b, 0)$ (or in $(0, b)$), where $a, b > 0$, then the zeros of $C(z)$ lie in $(-ab, ab)$.*
4. (Malo [64, p. 29], Schur [81]) *If the zeros of $p(z) = \sum_{k=0}^{\mu} a_k z^k$ are all real and if the zeros of $q(z) = \sum_{k=0}^{\nu} b_k z^k$ are all real and of the same sign, then the zeros of the polynomials $h(z) = \sum_{k=0}^{m} k! a_k b_k z^k$ and $f(z) = \sum_{k=0}^{m} a_k b_k z^k$ are also all real, where $m = \min(\mu, \nu)$.*

As a particularly interesting example of the last of these results, take $q(z) = (z+1)^\nu$ to see that for any positive integer ν, the polynomial $p(z)$ transforms to $\sum_{k=0}^{\mu} \binom{\nu}{k} a_k z^k$ with only real zeros. In [62], [64] and the references cited in these monographs the reader will find a number variations and generalizations of Theorem 2.4 (see also the more recent work of A. Aziz [6], [7] and Z. Rubinstein [76]).

Among the many related results, we wish to single out here Weisner's sectorial version of Theorem 2.3 [86]; that is, composition theorems for polynomials whose zeros lie in certain sectors. Weisner's proofs are based on the Gauss-Lucas Theorem and Laguerre's Separation Theorem. In [16], N. G. de Bruijn further extended Weisner's results and obtained an independent, geometric proof of a generalized Malo-Schur-Szegö Composition Theorem. We conclude this section with de Bruijn's result which deserves to be better known. The details of the proof given below are sufficiently different from de Bruijn's original proof to merit their inclusion here.

Let $S_\alpha = \{z \mid \theta_1 < \arg z < \theta_1 + \alpha\}$ denote an open sector with vertex at the origin and aperture $\alpha \le \pi$. Similarly, set $S_\beta = \{z \mid \theta_2 < \arg z < \theta_2 + \beta\}$. If $\alpha + \beta \le 2\pi$ we denote the "product" sector by $S_\alpha S_\beta$, where

$$S_\alpha S_\beta = \{w \in \mathbb{C} \mid w = w_1 w_2, \quad \text{where} \quad w_1 \in S_\alpha, w_2 \in S_\beta\}.$$

The sector $-S_\alpha$ is defined as $-S_\alpha = \{-w \in \mathbb{C} \mid w \in S_\alpha\}$. In the sequel we will denote the open left half-plane by

$$H_L = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\} \ . \tag{2.4}$$

Theorem 2.5 (Generalized Malo-Schur-Szegö Composition Theorem [16]) *Let $A(z) = \sum_{k=0}^{m} a_k z^k$ and $B(z) = \sum_{k=0}^{n} b_k z^k$, $a_m b_n \neq 0$, and let*

$$C(z) = \sum_{k=0}^{\nu} k! a_k b_k z^k, \quad \text{where} \quad \nu = \min(m, n). \tag{2.5}$$

If $A(z)$ has all its zeros in the sector S_α ($\alpha \leq \pi$) and if $B(z)$ has all its zeros in the sector S_β ($\beta \leq \pi$), then $C(z)$ has all its zeros in the sector $-S_\alpha S_\beta$.

Remark 2.6 (Rotational independence) We claim that it suffices to prove the theorem in the special case when each sector has $\theta = 0$ as its initial ray. Indeed, suppose that the zeros of $A(z)$ lie in S_α and the zeros of $B(z)$ lie in S_β, where S_α and S_β are defined above. Then the zeros of the polynomials $A(e^{i\theta_1} z)$ and $B(e^{i\theta_2} z)$ lie in $e^{-i\theta_1} S_\alpha$ and $e^{-i\theta_2} S_\beta$, respectively. In this case, by assumption, the zeros of the composite polynomial (which is now) $C(e^{i(\theta_1+\theta_2)} z)$ lie in the sector $-e^{-i(\theta_1+\theta_2)} S_\alpha S_\beta$. But then $C(z)$ has its zeros in $-S_\alpha S_\beta$, as desired. A similar argument shows that if the theorem holds for any particular S_α and S_β, then it holds for any rotations of those sectors.

Lemma 2.7 *Theorem 2.5 holds when S_α and S_β are half-planes.*

Proof. We consider the case when $\alpha = \beta = \pi$. By Remark 2.6, it suffices to prove that if all the zeros of $A(z)$ and $B(z)$ lie H_L, then $C(z)$ cannot vanish on the positive real axis. In order to prove this assertion, set

$$A(z) = a_m \prod_{j=1}^{m} (z - \alpha_j) \quad \text{and} \quad B(z) = b_n \prod_{j=1}^{n} (z - \beta_j),$$

where $\operatorname{Re}\, \alpha_j,\ \operatorname{Re}\, \beta_j < 0$, for all j. Fix $\lambda > 0$ and fix z with $x = \operatorname{Re}\, z \geq 0$. Then, logarithmic differentiation yields

$$\operatorname{Re}\left(\frac{A'(z)}{A(z)}\right) = \operatorname{Re}\left(\sum_{j=1}^{m} \frac{1}{z - \alpha_j}\right) = \sum_{j=1}^{m} \frac{x - \operatorname{Re} \alpha_j}{|z - \alpha_j|^2} > 0. \tag{2.6}$$

Thus, $A_1(z) := \lambda A'(z) - \beta_1 A(z) \neq 0$. (Indeed, if $A_1(z) = \lambda A'(z) - \beta_1 A(z) = 0$, then $A'(z)/A(z) = \beta_1/\lambda$. But then this would contradict (2.6), since

Re $\beta_1 < 0$.) Therefore, all the zeros of $A_1(z)$ lie in the open left half-plane H_L. By the same argument we see that all the zeros of

$$A_2(z) = \lambda A_1'(z) - \beta_2 A_1(z) = \lambda^2 A''(z) - \lambda(\beta_1 + \beta_2)A'(z) + \beta_1\beta_2 A(z)$$

lie in H_L. Continuing in this manner, we find that all the zeros of

$$\varphi(z) = \sum_{k=0}^{\nu} b_k \lambda^k A^{(k)}(z)$$

lie in the open left half-plane H_L. Thus, (cf. (2.3))

$$\varphi(0) = \sum_{k=0}^{\nu} b_k \lambda^k k! a_k = C(\lambda) \neq 0,$$

and, since $\lambda > 0$ was arbitrary, $C(z)$ does not vanish on the positive real axis. □

Proof of Theorem 2.5. By Lemma 2.7 and Remark 2.6, the theorem is true for half-planes. Let H_γ and H_δ be two half-planes, with initial rays $\theta = -\pi + \gamma$ and $\theta = -\pi + \delta$, respectively, and terminal rays $\theta = \gamma$ and $\theta = \delta$, respectively. Then all the zeros of $C(z)$ lie in $-H_\gamma H_\delta$; that is, they lie off the ray $\theta = \pi + \gamma + \delta$.

By Remark 2.6, it suffices to prove the result for sectors S_α and S_β whose initial rays lie on the positive x-axis. Thus, we have to show that all the zeros of $C(z)$ lie in $-S_\alpha S_\beta$, the open sector bounded by the rays $\theta = \pi$ and $\theta = \pi + \alpha + \beta$. We apply Lemma 2.7 for each $H_\gamma \supseteq S_\alpha,\ H_\delta \supseteq S_\beta$. Thus $\alpha \leq \gamma \leq \pi$ and $\beta \leq \delta \leq \pi$. Therefore, the zeros of $C(z)$ cannot lie on the rays in the closed sector from $\theta = \pi + \alpha + \beta$ to $\theta = \pi$. But this leaves all the zeros in $-S_\alpha S_\beta$. □

We observe that continuity considerations show that Theorem 2.5 remains valid when the open sectors are replaced by closed sectors, provided that we append the condition that the polynomial $C(z)$ is not identically zero. From Theorem 2.5 we can deduce several corollaries (cf. [16]). For example, if the zeros of the polynomial $A(z)$ all lie in the sector S_α ($\alpha \leq \pi$) and if the zeros of $B(z)$ are all real, then the zeros of $C(z)$ lie in $S_\alpha \cup -S_\alpha$. This follows from two applications of Theorem 2.5: First let S_β represent the closed upper half-plane, and then let S_β represent the closed lower half-plane. In particular, the Malo–Schur result (see part (4) of Theorem 2.4) is a special case of this, where $B(z)$ has only real zeros and the zeros of $A(z)$ are all real and of the same sign.

3. Multiplier sequences and the Laguerre-Pólya class

It follows from part (4) of Theorem 2.4 that if the polynomial $\sum_{k=0}^{n} b_k z^k$ has only real negative zeros, then the sequence $T = \{b_k\}_{k=0}^{\infty}$ is a multiplier sequence, where $b_k = 0$ if $k > n$ (see Definition 1.4). In this section we state several necessary and sufficient conditions for a sequence to be a multiplier sequence. The transcendental characterization of these sequences is given in terms of functions in the Laguerre-Pólya class (see Definition 3.1), while the algebraic characterization rests on properties of a class of polynomials called Jensen polynomials (Definition 3.4). In addition, we discuss a number of topics related to multiplier sequences and functions in the Laguerre-Pólya class: the closure properties of functions in the Laguerre-Pólya class, the Turán and Laguerre inequalities, the complex analog of the Laguerre inequalities, iterated Turán and Laguerre inequalities, the connection between totally positive sequences and multiplier sequences, the Gauss-Lucas property and convexity properties of increasing multiplier sequences, the Pólya–Wiman Theorem and the Fourier–Pólya Theorem, the Pólya–Wiman Theorem and certain differential operators and several open problems (including a problem due to Gauss).

Definition 3.1 A real entire function $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to be in the *Laguerre-Pólya class*, $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, if $\varphi(x)$ can be expressed in the form

$$\varphi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}}, \tag{3.1}$$

where $c, \beta, x_k \in \mathbb{R}$, $c \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and the sum $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. If $-\infty \leq a < b \leq \infty$ and if $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ has all its zeros in (a,b) (or $[a,b]$), then we will use the notation $\varphi \in \mathcal{L}\text{-}\mathcal{P}(a,b)$ (or $\varphi \in \mathcal{L}\text{-}\mathcal{P}[a,b]$). If $\gamma_k \geq 0$ (or $(-1)^k \gamma_k \geq 0$ or $-\gamma_k \geq 0$) for all $k = 0, 1, 2 \ldots$, then $\varphi \in \mathcal{L}\text{-}\mathcal{P}$ is said to be of *type I in the Laguerre-Pólya class*, and we will write $\varphi \in \mathcal{L}\text{-}\mathcal{P}I$. We will also write $\varphi \in \mathcal{L}\text{-}\mathcal{P}^{+}$, if $\varphi \in \mathcal{L}\text{-}\mathcal{P}I$ and $\gamma_k \geq 0$ for all $k = 0, 1, 2 \ldots$.

In order to clarify the above terminology, we remark that if $\varphi \in \mathcal{L}\text{-}\mathcal{P}I$, then $\varphi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$ or $\varphi \in \mathcal{L}\text{-}\mathcal{P}[0, \infty)$, but that an entire function in $\mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$ need not belong to $\mathcal{L}\text{-}\mathcal{P}I$. Indeed, if $\varphi(x) = \frac{1}{\Gamma(x)}$, where $\Gamma(x)$ denotes the gamma function, then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$, but $\varphi(x) \notin \mathcal{L}\text{-}\mathcal{P}I$. This can be seen, for example, by looking at the Taylor coefficients of $\varphi(x) = \frac{1}{\Gamma(x)}$.

Remark 3.2 (a) The significance of the Laguerre-Pólya class in the theory of entire functions stems from the fact that functions in this class, *and only these*, are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros (Levin [60, Chapter VIII]). Thus it follows that the Laguerre-Pólya class is closed under differentiation; that is, if $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, then $\varphi^{(n)}(x) \in \mathcal{L}\text{-}\mathcal{P}$ for $n = 0, 1, 2 \ldots$. In fact a more general closure property is valid. Indeed, let $D := \dfrac{d}{dx}$ denote differentiation with respect to x and suppose that the entire functions $\varphi(x) = \displaystyle\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ and $\psi(x)$ are in $\mathcal{L}\text{-}\mathcal{P}$. If the action of the differential operator $\varphi(D)$ is defined by

$$\varphi(D)\psi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} \psi^{(k)}(x), \tag{3.2}$$

and if the right-hand side of (3.2) represents an entire function, then the function $\varphi(D)\psi(x) \in \mathcal{L}\text{-}\mathcal{P}$. An analysis of various types of infinite order differential operators acting on functions in $\mathcal{L}\text{-}\mathcal{P}$ is carried out in [23].

(b) To further underscore the importance of the Laguerre-Pólya class, we cite here a few selected items from the extensive literature dealing with the differential operator $\varphi(D)$, where $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$. In connection with the study of the distribution of zeros of certain Fourier transforms, Pólya characterized the universal factors ([68] or [72, pp. 265–277]) in terms of $\varphi(D)$, where $\varphi \in \mathcal{L}\text{-}\mathcal{P}$. Subsequently, this work of Pólya was extended by de Bruijn [17] who studied, in particular, the operators $\cos(\lambda D)$ and $e^{-\lambda D^2}$, $\lambda > 0$. Benz [11] applied the operator $1/\varphi(D)$, $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, to investigate the distribution of zeros of certain exponential polynomials. The operators $\varphi(D)$, $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, play a central role in Schoenberg's celebrated work [79] on Pólya frequency functions and totally positive functions. Hirschman and Widder [44] used $\varphi(D)$, $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, to develop the inversion and representation theories of certain convolution transforms. More recently, Boas and Prather [13] considered the final set problem for certain trigonometric polynomials when differentiation D is replaced by $\varphi(D)$.

Theorem 3.3 ([73], [60, Chapter VIII], [64, Kapitel II]) *Let* $T = \{\gamma_k\}_{k=0}^{\infty}$, *where* $\gamma_k \geq 0$ *for* $k = 0, 1, 2 \ldots$.

1. (Transcendental Characterization.) T *is a multiplier sequence if and only if*

$$\varphi(x) = T[e^x] := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+. \tag{3.3}$$

2. (Algebraic Characterization.) T *is a multiplier sequence if and only if*

$$g_n(x) := T[(1+x)^n] := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j \in \mathcal{L}\text{-}\mathcal{P}^+ \quad \textit{for all } n = 1, 2, 3 \ldots. \tag{3.4}$$

We remark that the Taylor coefficients of functions in the Laguerre-Pólya class have analogous characterizations. It is the sign regularity property of the Taylor coefficients $\{\gamma_k\}_{k=0}^{\infty}$ of a function in $\mathcal{L}$-$\mathcal{P}I$ (that is, the terms γ_k all have the same sign or they alternate in sign) that allows us to invoke the Malo-Schur Composition Theorem (part (4) of Theorem 2.4) and thus deduce the remarkable algebraic characterization (3.4) of multiplier sequences.

Definition 3.4 Let $f(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ be an arbitrary entire function. Then the *nth Jensen polynomial* associated with the entire function $f^{(p)}(x)$ is defined by

$$g_{n,p}(x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_{j+p} x^j \qquad (n, p = 0, 1, 2, \dots). \tag{3.5}$$

If $p = 0$, we will write $g_{n,0}(x) = g_n(x)$.

The Jensen polynomials associated with arbitrary entire functions enjoy a number of important properties (cf. [22], [34]). For example, the sequence $\{g_{n,p}(t)\}_{n=0}^{\infty}$ is generated by $e^x f^{(p)}(xt)$, that is,

$$e^x f^{(p)}(xt) = \sum_{n=0}^{\infty} g_{n,p}(t) \frac{x^n}{n!}, \quad p = 0, 1, 2, \dots. \tag{3.6}$$

Moreover, it is not difficult to show that for $p = 0, 1, 2, \dots$,

$$\lim_{n\to\infty} g_{n,p}\left(\frac{z}{n}\right) = f^{(p)}(z),$$

holds uniformly on compact subsets of $\mathbb{C}$ [22, Lemma 2.2]. Observe that, if $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}$-$\mathcal{P}$, then $\varphi(D)x^n = g_n^*(x)$ for each n, $n = 0, 1, 2, \dots$, where the polynomials $g_n^*(x)$, called *Appell polynomials* (Rainville [75, p. 145]), are defined by $g_n^*(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^{n-k}$. If $\{g_n(t)\}_{n=0}^{\infty}$ is a sequence of Jensen polynomials associated with a function $\varphi \in \mathcal{L}$-$\mathcal{P}^+$, then it follows from the generating relation (3.6) that the sequence $\{g_n(t)\}_{n=0}^{\infty}$ is itself a multiplier sequence for each fixed $t \geq 0$.

We next consider several necessary and sufficient conditions for a real entire function

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \tag{3.7}$$

to belong to the Laguerre-Pólya class.

Theorem 3.5 ([22, Corollary 2.6]) *Let $\varphi(x)$ be an entire function defined by (3.7). Let*

$$\Delta_n(t) = g_n(t)^2 - g_{n-1}(t)g_{n+1}(t) \qquad (n = 1, 2, 3, \ldots;\ t \in \mathbb{R}), \tag{3.8}$$

where $g_n(t)$ is the nth Jensen polynomial associated with $\varphi(x)$. Suppose that $\gamma_k \neq 0$ for $k = 0, 1, 2 \ldots$. Then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ if and only if

$$\Delta_n(t) > 0 \quad \textit{for all real} \quad t \neq 0 \quad \textit{and}$$

$$\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} > 0 \quad (n = 1, 2, 3, \ldots). \tag{3.9}$$

In particular, if $\gamma_k > 0$ for $k = 0, 1, 2 \ldots$, then the sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is multiplier sequence if and only if (3.9) holds.

In [22, Theorem 2.5, Corollary 2.6, Theorem 2.7] the reader will find other formulations of Theorem 3.5 expressed in terms of Jensen polynomials. In order to state a different type of characterization of functions in $\mathcal{L}\text{-}\mathcal{P}$, we consider, for each fixed $x \in \mathbb{R}$, the Taylor series expansion of $\varphi(x + iy)\varphi(x - iy)$, where φ is a real entire function. Then an elementary calculation shows (cf. [28, Remark 2.4]) that, for each fixed $x \in \mathbb{R}$,

$$|\varphi(x + iy)|^2 = \varphi(x + iy)\varphi(x - iy) = \sum_{n=0}^{\infty} L_n(\varphi(x))y^{2n},$$

where $L_n(\varphi(x))$ is given by the formula

$$L_n(\varphi(x)) = \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!} \binom{2n}{j} \varphi^{(j)}(x)\varphi^{(2n-j)}(x). \tag{3.10}$$

Theorem 3.6 ([65], [34, Theorem 2.9], [28, Theorem 2.2]) *Let $\varphi(x)$, $\varphi(x) \not\equiv 0$, be a real entire function whose Taylor series expansion is given by (3.7). Suppose that $\varphi(x) = e^{-\alpha x^2}\varphi_1(x)$, where $\alpha \geq 0$ and the genus of $\varphi_1(x)$ is 0 or 1. Then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ if and only if $L_n(\varphi) \geq 0$ for all $n = 0, 1, 2, \ldots$. In particular, if $\gamma_k > 0$ for $k = 0, 1, 2, \ldots$, then the sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if $L_n(\varphi) \geq 0$ for all n.*

Since the Laguerre-Pólya class is closed under differentiation (cf. Remarks 3.2 (a)), it follows from Theorem 3.6 that $L_n(\varphi^{(k)}(x)) \geq 0$ for all $n, k = 0, 1, 2, \ldots$ and for all $x \in \mathbb{R}$. By specializing to the case when $n = 1$ we obtain the following necessary conditions for $\varphi(x)$ to belong to $\mathcal{L}\text{-}\mathcal{P}$.

Corollary 3.7 *Let $\varphi(x)$ be an entire function defined by (3.7). If $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, then the following inequalities hold.*

1. (The Turán Inequalities [22].)

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \quad k = 0, 1, 2 \ldots. \tag{3.11}$$

2. (The Laguerre Inequalities [22].)

$$L_1(\varphi^{(k)}(x)) = \left(\varphi^{(k+1)}(x)\right)^2 - \varphi^{(k)}(x)\varphi^{(k+2)}(x) \geq 0, \tag{3.12}$$

$$k = 0, 1, 2 \ldots, \quad x \in \mathbb{R}.$$

While the Turán and Laguerre inequalities are some of the simplest conditions that a function in $\mathcal{L}$-$\mathcal{P}$ must satisfy, the verification of the Laguerre inequalities, in general, is a nontrivial matter. For higher order inequalities of the type (3.12) see S. Karlin and G. Szegö [51]. Other extensions and applications may be found in M. Patrick [65] and H. Skovgaard [84]. We next proceed to describe various ramifications, extensions, generalizations and open problems related to these fundamental, albeit basic, inequalities. First, we note that there is a complex analog of the Laguerre inequalities which, in conjunction with appropriate growth conditions, characterizes functions in $\mathcal{L}$-$\mathcal{P}$.

Theorem 3.8 (Complex Laguerre Inequalities [34, Theorem 2.10]) *If a real entire function $\varphi(x)$, $\varphi(x) \not\equiv 0$, has the form $\varphi(x) = e^{-\alpha x^2}\varphi_1(x)$, where $\alpha \geq 0$ and the genus of $\varphi_1(x)$ is 0 or 1, then $\varphi(x) \in \mathcal{L}$-$\mathcal{P}$ if and only if*

$$|\varphi'(z)|^2 \geq \operatorname{Re}\left(\varphi(z)\overline{\varphi''(z)}\right) \quad \textit{for all} \quad z \in \mathbb{C}. \tag{3.13}$$

Is there a real variable analog of Theorem 3.8? That is, can the Laguerre inequalities (3.12) be strengthened with some supplementary hypotheses to yield a sufficient condition? To shed light on this question, for each $t \in \mathbb{R}$ we associate with a real entire function $f(x)$, the real entire function

$$f_t(x) := f(x + it) + f(x - it). \tag{3.14}$$

Now it is not difficult to show that $f_t(x) = 2\cos(tD)f(x)$, where $D = d/dx$. Also, if $f \in \mathcal{L}$-$\mathcal{P}$, then it follows from an extension of the Hermite-Poulain Theorem ([67, §3] or [72, p. 142]) that $f_t \in \mathcal{L}$-$\mathcal{P}$ for all $t \in \mathbb{R}$ and so by Corollary 3.7, $L_1(f_t(x)) \geq 0$ for all $x \in \mathbb{R}$. If $f \in \mathcal{L}$-$\mathcal{P}$ and if f is not of the form $C\exp(bx)$, $C, b \in \mathbb{R}$, then it is known that $L_1(f_t(x)) > 0$ for all $x, t \in \mathbb{R}$, $t \neq 0$ [32, Theorem I]. The main results in [32] are converses of this implication under some additional assumptions on the distribution of zeros of f. The proofs involve the study of the level sets of f, that is, the sets

$$\{z \in \mathbb{C} \mid \operatorname{Re}(e^{i\theta} f(z)) = 0\}, \quad \theta \in \mathbb{R}.$$

The analysis of the connections between the Laguerre expression $L_1(f_t)$ of f_t, the level set $\mathrm{Re}\, f = 0$ and the zero set of f_t is the dominant theme of this paper. Also in this paper the authors state that they "do not know if the converse of Theorem I (cited above) is valid in the absence of additional assumptions" [32, p. 379]. Here we note that the strict inequality $L_1(f_t(x)) > 0$, for all $x, t \in \mathbb{R}$, $t \neq 0$, is necessary as the following example shows. Let $f(x) = x(1 + x^2)$. Then an elementary, but tedious, calculation shows that

$$\begin{aligned} L_1(f_t(x)) &= 4\,(1 - 6t^2 + 9t^4 + 3x^4) \\ &= 4((1 - 3t^2)^2 + 3x^4) \geq 0 \end{aligned}$$

and equals 0 only if $x = 0$ and $t = \pm 1/\sqrt{3}$. Thus $L_1(f_t(x)) \geq 0$ for all $x, t \in \mathbb{R}$, but $f \notin \mathcal{L}\text{-}\mathcal{P}$. If we replace the differential operator $\cos(tD)$ by $\varphi(tD)$, where $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, then we are led to the following problem.

Problem 3.9 Let f be a real entire function of order less than 2. Suppose that

$$L_1(\varphi(tD) f(x)) > 0 \quad \text{for all} \quad x, t \in \mathbb{R},\ t \neq 0, \quad \text{and for all} \quad \varphi \in \mathcal{L}\text{-}\mathcal{P}. \tag{3.15}$$

If (3.15) holds, is $f \in \mathcal{L}\text{-}\mathcal{P}$? (See [23, p. 806] for the reasons for this restriction on the growth of f.)

We next explore some other avenues that might provide stronger necessary conditions than those stated in Corollary 3.7. To this end, we consider iterating the Laguerre and Turán inequalities.

Definition 3.10 For any real entire function $\varphi(x)$, set

$$\mathcal{T}_k^{(1)}(\varphi(x)) := (\varphi^{(k)}(x))^2 - \varphi^{(k-1)}(x)\varphi^{(k+1)}(x) \quad \text{if} \quad k \geq 1,$$

and for $n \geq 1$, set

$$\mathcal{T}_k^{(n)}(\varphi(x)) := (\mathcal{T}_k^{(n-1)}(\varphi(x)))^2 - \mathcal{T}_{k-1}^{(n-1)}(\varphi(x))\, \mathcal{T}_{k+1}^{(n-1)}(\varphi(x)) \quad \text{if } k \geq n.$$

Note that with the above notation, we have $\mathcal{T}_{k+j}^{(n)}(\varphi) = \mathcal{T}_k^{(n)}(\varphi^{(j)})$ for $k \geq n$ and $j = 0, 1, 2 \ldots$, and that $L_1(\varphi^{(k-1)}(x)) = \mathcal{T}_k^{(1)}(\varphi(x))$ for $k \geq 1$. The authors' earlier investigations of functions in the Laguerre-Pólya class [22], [25], [28] have led to the following open problem.

Problem 3.11 ([28, §3]) If $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, are the iterated Laguerre inequalities valid for all $x \geq 0$? That is, is it true that

$$\mathcal{T}_k^{(n)}(\varphi(x)) \geq 0 \qquad \text{for all} \quad x \geq 0 \quad \text{and} \quad k \geq n? \tag{3.16}$$

In the formulation of Problem 3.11, the restriction to the class $\mathcal{L}\text{-}\mathcal{P}^+$ is necessary, since simple examples show that (3.16) need not hold for functions in $\mathcal{L}\text{-}\mathcal{P} \setminus \mathcal{L}\text{-}\mathcal{P}^+$. For example, $\varphi(x) = (x-2)(x+1)^2 \in \mathcal{L}\text{-}\mathcal{P}$, but a calculation shows that $\mathcal{T}_2^{(2)}(\varphi(x))$ is negative for all sufficiently small positive values of x. In [22, Theorem 2.13] the authors have shown that (3.16) is true when $n = 2$; that is, the double Laguerre inequalities are valid. The proof there is based on certain polynomial invariants and Theorem 3.6. A somewhat shorter proof, which also depends on Theorem 3.6 is given in [28, Theorem 3.5].

Theorem 3.12 ([22, Theorem 2.13], [28, Theorem 3.5]) *If* $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, *then for* $j = 0, 1, 2 \ldots,$

$$\mathcal{T}_k^{(2)}(\varphi^{(j)}(x)) \geq 0 \qquad \textit{for all} \quad x \geq 0 \quad \textit{and} \quad k \geq 2. \tag{3.17}$$

A particularly intriguing open problem arises in the special case when $\varphi(x) = x^m$ $(m = 1, 2, 3 \ldots)$ in (3.16).

Problem 3.13 ([28, §3]) Is it true that

$$\mathcal{T}_n^{(n)}(x^{n+k}) \geq 0 \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad k, n = 1, 2, 3 \ldots ? \tag{3.18}$$

We next turn to the iterated Turán inequalities.

Definition 3.14 Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. We define the r-th *iterated Turán sequence* of Γ via $\gamma_k^{(0)} := \gamma_k$, $k = 0, \ldots,$ and $\gamma_k^{(r)} := (\gamma_k^{(r-1)})^2 - \gamma_{k-1}^{(r-1)}\gamma_{k+1}^{(r-1)}$, $k = r, r+1, \ldots.$

Thus, if we write $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$, then $\gamma_k^{(r)}$ is just $\mathcal{T}_k^{(r)}(\varphi(x))$ evaluated at $x = 0$. In [28, §4] the authors have shown that for multiplier sequences which decay sufficiently rapidly *all* the higher iterated Turán inequalities hold. The main result of [28] is that the third iterated Turán inequalities are valid for all functions of the form $\varphi(x) = x^2\psi(x)$, where $\psi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$.

Theorem 3.15 ([28, Theorem 5.5]) *Let* $\psi(x) := \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+$ *and set*

$$\varphi(x) := x^2\psi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k,$$

so that $\gamma_0 = \gamma_1 = 0$ *and* $\gamma_k = k(k-1)\alpha_{k-2}$, *for* $k = 2, 3, \ldots.$ *Then*

$$\gamma_3^{(3)} = \left(\mathcal{T}_3^{(3)}(\varphi(x))\right)_{x=0} \geq 0. \tag{3.19}$$

An examination of the proof of Theorem 3.15 shows that the restriction that $\varphi(x)$ has a double zero at the origin is merely a ploy to render the, otherwise very lengthy and involved, computations tractable.

We next touch upon the characterization of entire functions in $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ purely in terms of their Taylor coefficients. To this end, we consider the entire function

$$\varphi(x) := \sum_{k=0}^{\infty} \alpha_k x^k \quad \text{where} \quad \alpha_k = \frac{\gamma_k}{k!}, \quad \gamma_0 = 1, \quad \gamma_k \geq 0 \quad (k = 1, 2, 3 \dots). \tag{3.20}$$

and recall the following definition.

Definition 3.16 A real sequence $\{\alpha_k\}_{k=0}^{\infty}$, $\alpha_0 = 1$, is said to be a *totally positive sequence*, if the infinite lower triangular matrix

$$A = (\alpha_{i-j}) = \begin{pmatrix} \alpha_0 & 0 & 0 & 0 & 0 & \dots \\ \alpha_1 & \alpha_0 & 0 & 0 & 0 & \dots \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \dots \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots \\ & & \dots & & & \end{pmatrix} \quad (i, j = 1, 2, 3, \dots), \tag{3.21}$$

is totally positive; that is, all the minors of A of all orders are nonnegative.

In [1, p. 306], M. Aissen, A. Edrei, I. J. Schoenberg and A. Whitney characterized the generating functions of totally positive sequences. A special case of their result is the following theorem.

Theorem 3.17 ([1, p. 306]) *Let $\varphi(x)$ be the entire function defined by (3.20). Then $\{\alpha_k\}_{k=0}^{\infty}$ is a totally positive sequence if and only if $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$.*

An immediate consequence of Theorem 3.17 is the following corollary.

Corollary 3.18 ([1, p. 306]) *Let*

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n \qquad (\alpha_0 = 1,\ \alpha_k \geq 0,\ k = 1, \dots, n).$$

Then $p(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ if and only if the sequence $\alpha_0, \alpha_1, \dots, \alpha_n, 0, 0, \dots$ is a totally positive sequence.

Suppose that the generating function (3.20) is an entire function. Then, in light of Theorem 3.3(1), the sequence $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_0 = 1$, $\gamma_k \geq 0$, $k = 0, 1, 2, \dots$, is a multiplier sequence if and only if the sequence $\{\frac{\gamma_k}{k!}\}_{k=0}^{\infty}$ is a totally positive sequence.

Remarks. Totally positive sequences were first introduced in 1912 by M. Fekete and G. Pólya [36]. For a concise survey of totally positive matrices we refer to T. Ando [4]. The connection between totally positive sequences and combinatorics is treated in F. Brenti's monograph [15]. S. Karlin's monumental tome [50] on total positivity, while mostly concerned with totally positive kernels, also treats totally positive matrices and I. J. Schoenberg's theory of variation diminishing transformations [78]. From the extensive literature treating total positivity and related topics, here we merely mention the recent work of M. Alzugaray [3] and O. M. Katkova and I. V. Ostrovskiĭ[52] investigating the zero sets of generating functions of multiply positive sequences. (These are sequences which have the property that the minors of the Toeplitz matrix (3.21), less than or equal to some fixed order, are all nonnegative.)

Increasing multiplier sequences enjoy a number interesting geometric properties some of which we now proceed to sketch here. To facilitate our description, we introduce the following terminology.

Definition 3.19 A sequence of real numbers $T = \{\beta_k\}_{k=0}^{\infty}$ is said to possess the *Gauss-Lucas property*, if whenever a convex region K contains the origin and all the zeros of a complex polynomial $f(z) = \sum_{k=0}^{n} a_k z^k$, then all the zeros of the polynomial $T[f(z)] = \sum_{k=0}^{n} \beta_k a_k z^k$ also lie in K.

The proof of the complete characterization of sequences which enjoy the Gauss-Lucas property hinges on the Malo-Schur-Szegö Composition Theorem (cf. Theorem 2.4(1)) and on the fact that the zeros of the Jensen polynomials associated with an increasing multiplier sequence must all lie in the interval $[0, 1]$ (see [20, Theorem 2.3]).

Theorem 3.20 ([20, Theorem 2.8]) *Let* $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \geq 0$, *be a nonzero sequence of real numbers. Then* T *possess the Gauss-Lucas property if and only if* T *is a multiplier sequence and* $0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$.

The classical example of this theorem is its application to the sequence $T = \{0, 1, 2, \dots\}$, since $T[f(x)] = xf'(x)$ for any polynomial f. This, and other examples, suggest that the operators T may be viewed as generalized forms of differential operators. The problem of extending the foregoing results to transcendental entire functions whose zeros lie in an unbounded convex region appears to be very difficult. However, for transcendental entire functions of genus zero, we have the following consequence of Theorem 3.20.

Corollary 3.21 ([20, Corollary 3.1]) *Let* $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \geq 0$, *be an increasing multiplier sequence. Let* K *be an unbounded convex region which contains the origin and all the zeros of an entire function* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *of genus zero. Then the zeros of the entire function* $T[f(z)] = \sum_{k=0}^{\infty} \gamma_k a_k z^k$ *also lie in* K.

We remark that the extension of these results to real entire functions of less restricted growth, but all whose zeros are real, is still open (cf. S. Hellerstein and J. Korevaar [42]).

Turning to the convexity properties of multiplier sequences, we first note that all multiplier sequences $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \geq 0$, are eventually monotone; that is, from a certain point onward the multiplier sequence T is either increasing or decreasing (cf. [20, Proposition 4.4]). In the sequel it will be convenient for us to adopt the following standard notation (the Δ-notation) for forward differences. (A caveat is in order. The symbol $\Delta_n(t)$ used in (3.8) has a different meaning.)

Definition 3.22 For any real sequence $\{\gamma_k\}_{k=0}^{\infty}$, we define $\Delta^0\gamma_p := \gamma_p$, $\Delta\gamma_p := \gamma_{p+1} - \gamma_p$ and

$$\Delta^n\gamma_p := \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j}\gamma_{p+j} \quad \text{for} \quad n,\, p = 0, 1, 2, \ldots. \tag{3.22}$$

Proposition 3.23 ([20, Proposition 4.2]) *Let* $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+$. *If* $0 \leq \gamma_0 \leq \gamma_1 \leq \cdots$, *then*

$$\Delta^n\gamma_p \geq 0 \quad \textit{for} \quad n, p = 0, 1, 2, \ldots. \tag{3.23}$$

Moreover,

$$\sum_{n=0}^{\infty} \frac{\Delta^n\gamma_p}{n!} x^n \in \mathcal{L}\text{-}\mathcal{P}^+ \quad \textit{for} \quad p = 0, 1, 2, \ldots.$$

Since the Laguerre-Pólya class is closed under differentiation, it follows that if $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+$, then

$$\psi_n(x) = \sum_{p=0}^{\infty} \frac{\Delta^n\gamma_p}{p!} x^p = e^x \frac{d^n}{dx^n} e^{-x}\varphi(x) \in \mathcal{L}\text{-}\mathcal{P} \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{3.24}$$

If we assume that $0 \leq \gamma_0 \leq \gamma_1 \leq \cdots$, then $\{\Delta^n\gamma_p\}_{p=0}^{\infty}$ is also an increasing sequence by Proposition 3.23 and thus we conclude that $\psi_n(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ for each fixed nonnegative integer n. Now for $n = 2$, inequality (3.23) says that $\{\gamma_k\}_{k=0}^{\infty}$ is convex.

We conclude this section with a few remarks concerning two famous conjectures, related to functions in the Laguerre-Pólya class, which have been recently solved. These long-standing open problems, known in the literature as the Pólya–Wiman conjecture and the Fourier–Pólya conjecture, have been investigated by many eminent mathematicians. The history associated with these problems is particularly interesting. (See, for example, G. Pólya [71] or [72, pp. 394–407] for a general discussion of the theme, and

[70] or [72, pp. 322–335] for a comprehensive survey which covers almost everything in this area up to 1942.) The Pólya–Wiman conjecture has been established by T. Craven, G. Csordas, W. Smith [29], [30], Y.-O. Kim [54], [55]. We shall refer to their result as the Pólya–Wiman Theorem. Recently, H. Ki and Y.-O. Kim [53] provided a truly elegant proof of this theorem.

Theorem 3.24 (The Pólya–Wiman Theorem [29], [30], [55], [54], [53]) *Let* $f(x) = \exp(-\alpha x^2)g(x)$ *be a real entire function, where* $\alpha \geq 0$ *and suppose that the genus of* $g(x)$ *is at most 1. If* $f(x)$ *has only a finite number of nonreal zeros, then its successive derivatives, from a certain one onward, have only real zeros, that is* $D^m f(x) \in \mathcal{L}\text{-}\mathcal{P}$, $D = d/dx$, *for all sufficiently large positive integers* m.

Theorem 3.24 confirms the heuristic principle according to which the nonreal zeros of the derivatives $f^{(n)}(z)$ of a real entire function move toward the real axis when the order of $f(z)$ is less than 2. The dual principle asserts that the nonreal zeros of the derivatives $f^{(n)}(z)$ move away from the real axis when the order of $f(z)$ is greater than 2. A long-standing open problem related to this dual principle may be stated as follows. If the order of a real entire function $f(z)$ is greater than 2, and if $f(z)$ has only a finite number of nonreal zeros, then the number of the nonreal zeros of $f^{(n)}$ tends to infinity as $n \to \infty$ (G. Pólya [71]). Significant contributions to this problem were made by B. Ja. Levin and I.V. Ostrovskiĭ [61] and extended by S. Hellerstein and C. C. Yang [43]. In particular, S. Hellerstein and C. C. Yang showed that the conjecture is true for real entire functions of sufficiently large order (see also T. Sheil-Small [82]).

The Fourier–Pólya conjecture (established in [53]) asserts that one can determine the number of nonreal zeros of a real entire function $f(z)$ of genus 0 by counting the number of *critical points* of $f(z)$; $f(z)$ has just as many critical points as couples of nonreal zeros. When $f(z)$ and all its derivatives possess only simple zeros, then the critical points of $f(z)$ are the abscissae of points where $f^{(n)}(z)$ has positive minima or negative maxima. The definition of critical points is more elaborate if there are multiple zeros ([53], see also Y.-O. Kim [56], [57]).

We next consider a few sample results which pertain to investigations related to Theorem 3.24. In [23] the authors analyze the more general situation when the operator D in the Pólya–Wiman Theorem is replaced by the differential operator $\varphi(D)$, where $\varphi(x)$ need not belong to $\mathcal{L}\text{-}\mathcal{P}$. Indeed, if $f(x)$ is a real power series with zero linear term and if $p(x)$ is any real polynomial, then $[f(D)]^m p(x) \in \mathcal{L}\text{-}\mathcal{P}$ for all sufficiently large positive integers m. More precisely, the authors proved the following result.

Theorem 3.25 ([23, Theorem 2.4]) *Let*

$$f(x)=\sum_{k=0}^{\infty}\alpha_k\frac{x^k}{k!} \tag{3.25}$$

be a real power series with $\alpha_0 = 1$, $\alpha_1 = 0$ *and* $\alpha_2 < 0$. *Let* $p(x) = \sum_{k=0}^{n} a_k x^k$ *be any real polynomial of degree at least one. Then there is a positive integer* m_0 *such that* $[f(D)]^m p(x) \in \mathcal{L}\text{-}\mathcal{P}$ *for all* $m \geq m_0$. *In fact,* m_0 *can be chosen so that all the zeros are simple.*

If the linear term in (3.25) is nonzero, then simple examples show that the conclusion of Theorem 3.25 does not hold without much stronger restrictions on f [23, §3]. To rectify this, the authors consider $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ and a real entire function $f(x)$ having only a finite number of nonreal zeros (with some restriction of the growth of φ or f as in Theorem 3.26 below). If $\varphi(x)$ has at least one real zero, then $[\varphi(D)]^m f(x) \in \mathcal{L}\text{-}\mathcal{P}$ for all sufficiently large positive integers m. The proof of the following theorem is based on several technical results ([23, Lemma 3.1, Lemma 3.2] and [83, p. 41 and p. 106]) involving differential operators.

Theorem 3.26 ([23, Theorem 3.3]) *Let* φ_1 *and* f_1 *be real entire functions of genus* 0 *or* 1 *and set* $\varphi(x) = e^{-\alpha_1 x^2}\varphi_1(x)$ *and* $f(x) = e^{-\alpha_2 x^2} f_1(x)$, *where* $\alpha_1, \alpha_2 \geq 0$, $\alpha_1\alpha_2 = 0$. *If* $\varphi \in \mathcal{L}\text{-}\mathcal{P}$, f *has only a finite number of nonreal zeros and* $\varphi(x)$ *has at least one real zero, then there is a positive integer* m_0 *such that* $[\varphi(D)]^m f(x) \in \mathcal{L}\text{-}\mathcal{P}$ *for all* $m \geq m_0$.

A separate analysis of the operator $e^{-\alpha D^2}$, $\alpha > 0$, shows that, not only does Theorem 3.26 hold, but that the zeros become simple. In fact, if $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, where the order of φ is less than two, then $e^{-\alpha D^2}\varphi(x)$ has only real *simple* zeros.

Theorem 3.27 ([23, Theorem 3.10]) *Let* $f \in \mathcal{L}\text{-}\mathcal{P}$ *and suppose that the order of* f *is strictly less than* 2. *Let* $u(x,t) = e^{-tD^2} f(x)$ *for all* $t > 0$. *Then, for each fixed* $t > 0$, $u(x,t) \in \mathcal{L}\text{-}\mathcal{P}$ *and the zeros of* $u(x,t)$ *are all simple.*

Corollary 3.28 ([23, Theorem 3.11]) *Let* f *be a real entire function of order strictly less than* 2, *having only a finite number of nonreal zeros. If* $\alpha > 0$, *then* $[e^{-\alpha D^2}]^m f(x) \in \mathcal{L}\text{-}\mathcal{P}$ *with only simple zeros for all sufficiently large* m.

The question of simplicity of zeros is pursued further in [23, §4]. The authors proved that if $\varphi(x)$ and $f(x)$ are functions in the Laguerre-Pólya class of order less than two, φ has an infinite number of zeros, and there is a bound on the multiplicities of the zeros of f, then $\varphi(D)f(x)$ has only simple real zeros [23, Theorem 4.6]. In [23, p. 819] the question was raised whether or not the assumption (in [23, Theorem 4.6]) that there is a bound on the multiplicities of the zeros of f is necessary. That is, if $\varphi,\ f \in \mathcal{L}\text{-}\mathcal{P}$ and if φ has order less than two, then is it true that $\varphi(D)f(x)$ has only *simple* real zeros?

The study of the "movement" of the zeros under the action of the infinite order differential operators was initiated by G. Pólya ([67] or [72, pp. 128–153]) and N. G. de Bruijn [17] in their study of the distribution of zeros of entire functions related to the Riemann ξ-function. (For recent results in this direction see [33] and [31].) In [17], de Bruijn proved, in particular, that if f is a real entire function of order less than two and if all the zeros of f lie in the strip $S(d) := \{\, z \in \mathbb{C} \mid |\operatorname{Im} z| \leq d \,\}\quad (d \geq 0)$, then the zeros of $\cos(\lambda D)f(x)\quad(\lambda \geq 0)$ satisfy $|\operatorname{Im} z| \leq \sqrt{d^2 - \lambda^2}$ if $d > \lambda$, and $\operatorname{Im} z = 0$ if $0 \leq d \leq \lambda$. This result may be viewed as an analog of Jensen's theorem on the location of the nonreal zeros of the derivative of a polynomial [62, §7].

Problem 3.29 Is there also an analog of Jensen's theorem for $\varphi(\lambda D)f(x)$ when φ is an *arbitrary* function (not of the form $ce^{\beta x}$) in the Laguerre-Pólya class?

Finally, there is also an interesting connection between the ideas used to prove the Pólya–Wiman Theorem (for entire functions of order less than 2) [29, Theorem 1] and a question that was raised by Gauss in 1836 [29, p. 429]. Let $p(x)$ be a real polynomial of degree n, $n \geq 2$, and suppose that $p(x)$ has exactly $2d$ nonreal zeros, $0 \leq 2d \leq n$. Then Gauss' query is to find a relationship between the number $2d$ and the number of real zeros of the rational function

$$q(x) := \frac{d}{dx}\left(\frac{p'(x)}{p(x)}\right). \tag{3.26}$$

If $p(x)$ has only real zeros, then $q(x) < 0$ for all $x \in \mathbb{R}$, and consequently in this special case the answer is clear. Now it follows from [29, Theorem 1] that if for some $\lambda \in \mathbb{R}$, the polynomial $\lambda p(x) + p'(x)$ has only real zeros, then $q(x)$ has precisely $2d$ real zeros. On the basis of their analysis, the authors in [29, p. 429] stated the following conjecture.

Problem 3.30 Let $p(x)$ be a real polynomial of degree n, $n \geq 2$, and suppose that $p(x)$ has exactly $2d$ nonreal zeros, $0 \leq 2d \leq n$. Prove that

$$Z_R(q(x)) \leq 2d,$$

where $Z_R(q(x))$ denotes the number of real zeros, counting multiplicities, of the rational function $q(x)$ defined by (3.26).

Gauss' question has been studied by several authors (see the references in [29]). For recent contributions dealing with Problem 3.30 we refer to K. Dilcher and K. B. Stolarsky [35].

4. Complex zero decreasing sequences (CZDS)

It follows from Definition 1.5 that every complex zero decreasing sequence is also a multiplier sequence. If $T = \{\gamma_k\}_{k=0}^{\infty}$ is a sequence of *nonzero* real numbers, then inequality (1.5) is equivalent to the statement that for any polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, $T[p]$ has at least as many real zeros as p has. There are, however, CZDS which have zero terms and consequently it may happen that $\deg T[p] < \deg p$. When counting the real zeros of p, the number generally increases with the application of T, but may in fact decrease due to a decrease in the degree of the polynomial. For this reason, we count nonreal zeros rather than real ones. The existence of a *nontrivial* CZDS is a consequence of the following theorem proved by Laguerre and extended by Pólya ([69] or [72, pp. 314-321]). We remark that in the next theorem, part (2) follows from (1) by a limiting argument.

Theorem 4.1 (Laguerre [64, Satz 3.2])

1. *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitrary real polynomial of degree n and let $h(x)$ be a polynomial with only real zeros, none of which lie in the interval $(0, n)$. Then $Z_c(\sum_{k=0}^{n} h(k) a_k x^k) \leq Z_c(f(x))$.*
2. *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitrary real polynomial of degree n, let $\varphi \in \mathcal{L}\text{-}\mathcal{P}$ and suppose that none of the zeros of φ lie in the interval $(0, n)$. Then the inequality $Z_c(\sum_{k=0}^{n} \varphi(k) a_k x^k) \leq Z_c(f(x))$ holds.*
3. *Let $\varphi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$, then the sequence $\{\varphi(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence.*

As a particular example of a CZDS, we can apply Theorem 4.1(2) to the function $\frac{1}{\Gamma(x+1)} \in \mathcal{L}\text{-}\mathcal{P}$ to obtain $T = \{\frac{1}{k!}\}_{k=0}^{\infty}$. One of the main results of [24] is the converse of Theorem 4.1 in the case that φ is a polynomial. The converse fails, in general, for transcendental entire functions. Indeed, if $p(x)$ is a polynomial in $\mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$, then $\frac{1}{\Gamma(-x)} + p(x)$ and $\sin(\pi x) + p(x)$ are transcendental entire functions which generate the same sequence $\{p(k)\}_{k=0}^{\infty}$, but they are not in $\mathcal{L}\text{-}\mathcal{P}$. For several analogues and extensions of Theorem 4.1, we refer the reader to S. Karlin [50, pp. 379–383], M. Marden [62, pp. 60–74], N. Obreschkoff [64, pp. 6–8, 42–47]. A sequence $\{\gamma_k\}_{k=0}^{\infty}$ which can be interpolated by a function $\varphi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$, that is, $\varphi(k) = \gamma_k$ for $k = 0, 1, 2, \ldots$, will be called a *Laguerre multiplier sequence* or a *Laguerre sequence*. It follows from Theorem 4.1 that Laguerre sequences are multiplier sequences.

With the terminology adopted here, the Karlin-Laguerre problem [8], [24] can be formulated as follows.

Problem 4.2 (The Karlin-Laguerre problem.) Characterize all the multiplier sequences which are complex zero decreasing sequences (CZDS).

This fundamental problem in the theory of multiplier sequences has eluded the attempts of researchers for over four decades. In order to elucidate some of the subtleties involved, we need to introduce yet another family of sequences related to CZDS. The reciprocals of Laguerre sequences are examples of sequences which are termed in the literature as λ-*sequences* and are defined as follows (cf. L. Iliev [46, Ch. 4] or M. D. Kostova [58]).

Definition 4.3 A sequence of nonzero real numbers, $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$, is called a λ-*sequence* if

$$\Lambda[p(x)] = \Lambda\left[\sum_{k=0}^{n} a_k x^k\right] := \sum_{k=0}^{n} \lambda_k a_k x^k > 0 \text{ for all } x \in \mathbb{R}, \tag{4.1}$$

whenever $p(x) = \sum_{k=0}^{n} a_k x^k > 0$ for all $x \in \mathbb{R}$.

We remark that if Λ is a sequence of nonzero real numbers and if $\Lambda[e^{-x}]$ is an entire function, then a *necessary* condition for Λ to be a λ-sequence, is that $\Lambda[e^{-x}] \geq 0$ for all real x. (Indeed, if $\Lambda[e^{-x}] < 0$ for $x = x_0$, then continuity considerations show that there is a positive integer n such that $\Lambda[(1-\frac{x}{2n})^{2n} + \frac{1}{n}] < 0$ for $x = x_0$.)

In [46, Ch. 4] (see also [58]) it was pointed out by Iliev that λ-sequences are the positive semidefinite sequences. There are several known characterizations of positive definite sequences (see, for example, [63, Ch. 8] and [87, Ch. 3]) which we include here for the reader's convenience. See also [24, Theorem 1.7], where the first item should refer only to positive definite λ-sequence s.

Theorem 4.4 *Let* $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ *be a sequence of nonzero real numbers. Then the following are equivalent.*

1. (Positive Definite Sequences [87, p. 132]) *For any polynomial* $p(x) = \sum_{k=0}^{n} a_k x^k$, *$p$ not identically zero, the relation* $p(x) \geq 0$ *for all* $x \in \mathbb{R}$, *implies that*

$$\Lambda[p](1) = \sum_{k=0}^{n} \lambda_k a_k > 0.$$

2. (Determinant Criterion [87, p. 134])

$$\det(\lambda_{i+j}) = \begin{vmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_n \\ \lambda_1 & \lambda_2 & \dots & \lambda_{n+1} \\ \vdots & \vdots & & \vdots \\ \lambda_n & \lambda_{n+1} & \dots & \lambda_{2n} \end{vmatrix} > 0 \quad \textit{for } n = 0, 1, 2, \dots . \tag{4.2}$$

3. (The Hamburger Moment Problem [87, p. 134]) *There exists a non-decreasing function $\mu(t)$ with infinitely many points of increase such that*

$$\lambda_n = \int_{-\infty}^{\infty} t^n d\mu(t) \qquad \text{for } n = 0, 1, 2, \ldots. \tag{4.3}$$

The importance of λ-sequences in our investigation stems from the fact that a *necessary condition* for a sequence $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k > 0$, to be a CZDS is that the sequence of reciprocals $\Lambda = \{\frac{1}{\gamma_k}\}_{k=0}^{\infty}$ be a λ-sequence. Thus, for example, the reciprocal of a Laguerre multiplier sequence is a λ-sequence. As our next example shows, there are multiplier sequences whose reciprocals are not λ-sequences.

Example 4.5 ([24, p. 423]) Let $T = \{1 + k + k^2\}_{k=0}^{\infty}$. Then by Theorem 3.3, T is a multiplier sequence since

$$(1+x)^2 e^x = \sum_{k=0}^{\infty} \frac{1+k+k^2}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+.$$

Next, let $\Lambda = \{\lambda_k\}_{k=0}^{\infty} = \{\frac{1}{1+k+k^2}\}_{k=0}^{\infty}$. Then a calculation shows that the determinant $\det(\lambda_{i+j})$, $(i, j = 0, \ldots, 3)$, is

$$\begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{7} & \frac{1}{13} \\ \frac{1}{3} & \frac{1}{7} & \frac{1}{13} & \frac{1}{21} \\ \frac{1}{7} & \frac{1}{13} & \frac{1}{21} & \frac{1}{31} \\ \frac{1}{13} & \frac{1}{21} & \frac{1}{31} & \frac{1}{43} \end{vmatrix} = -\frac{55936}{2833723113403} = -1.9739\cdots \times 10^{-8}.$$

Therefore, by (4.2) we conclude that Λ is not a λ-sequence and *a fortiori* the multiplier sequence T is *not* a CZDS. It is also instructive to exhibit a concrete example for which inequality (1.5) fails. To this end, we set $p(x) := (x+1)^6(x^2 + \frac{1}{2}x + \frac{1}{5})$. Then a calculation shows that

$$T[p(x)] = \frac{1}{10}(x+1)^4(730x^4 + 785x^3 + 306x^2 + 43x + 2).$$

Now it can be verified that $Z_c(T[p(x)]) = 4 \not\leq Z_c(p(x)) = 2$, and hence again it follows that the multiplier sequence T is not a CZDS.

In light of Example 4.5, the following natural problem arises.

Problem 4.6 (Reciprocals of multiplier sequences.) Characterize the multiplier sequences $\{\gamma_k\}_{k=0}^{\infty}$ with $\gamma_k > 0$, for which the sequences of reciprocals, $\{1/\gamma_k\}_{k=0}^{\infty}$, are λ-sequences.

One of the principal results of [24, Theorem 2.13] characterizes the class of all polynomials which interpolate CZDS. The proof of the next theorem requires several preparatory results involving properties of both CZDS and λ-sequences.

Theorem 4.7 ([24, Theorem 2.13]) *Let $h(x)$ be a real polynomial. The sequence $T = \{h(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence (CZDS) if and only if either*

1. *$h(0) \neq 0$ and all the zeros of h are real and negative, or*
2. *$h(0) = 0$ and the polynomial $h(x)$ has the form*

$$h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{i=1}^{n}(x-b_i), \qquad (4.4)$$

where $b_i < m$ for each $i = 1, \ldots, n$ and m is a fixed positive integer.

We remark that in part (2) of Theorem 4.7, the assumption that $b_i < m$ for each $i = 1, \ldots, n$, is necessary. Indeed, set $m = 1$ and $n = 1$ in (4.4), so that $h(x) = x(x-b)$. If $b > 1$, then the sequence $T = \{h(k)\}_{k=0}^{\infty}$ has the form $0, 1-b, 2(2-b), 3(3-b), \ldots$, and thus the terms of the sequence eventually become positive even though $1 - b < 0$. It follows that T cannot even be a multiplier sequence. A similar claim can be made for sequences arising from polynomials of the form $x(x-1)(x-2)\cdots(x-m+1)(x-b)$ with $b > m$.

In general, if a sequence, $\{\gamma_k\}_{k=0}^{\infty}$, of positive real numbers grows sufficiently rapidly, then it is a λ-sequence. For example, recently the authors proved that if $\lambda_k > 0$, $\lambda_0 = 1$, and if $(4.07\cdots)\lambda_k^2 \leq \lambda_{k-1}\lambda_{k+1}$, then $\{\lambda_k\}_{k=0}^{\infty}$ is a positive definite sequence [27]. (The question whether or not the constant $4.07\cdots$ is best possible remains open.) Thus, applying this criterion to sequences of the form $\{e^{k^p}\}_{k=0}^{\infty}$, where p is a positive integer, $p \geq 3$, we see that such sequences are positive definite sequences. Furthermore, it is known that the sequence of reciprocals $\{e^{-k^p}\}_{k=0}^{\infty}$, (where p is a positive integer, $p \geq 3$) is a multiplier sequence [24, p. 438]. However, it is not known whether or not these multiplier sequences are CZDS. For ease of reference, and to tantalize the interested reader, we pose here the following concrete question.

Problem 4.8 (a) Is the sequence $\{e^{-k^3}\}_{k=0}^{\infty}$ a CZDS?

(b) More generally, if $\{\gamma_k\}_{k=0}^{\infty}$ is a positive multiplier sequence with the property that $\{1/\gamma_k\}_{k=0}^{\infty}$ is a λ-sequence, is it true that $\{\gamma_k\}_{k=0}^{\infty}$ is CZDS?

In order to establish the existence of additional classes of CZDS in [24, §4] the authors first generalized a classical theorem of Hutchinson [45] (see also Hardy [40] or [41, pp. 95-99], Petrovitch [66] and the recent paper by Kurtz [59, p. 259]) and obtained the following results.

Theorem 4.9 ([24, Theorem 4.3]) *Let $\varphi(x) = \sum_{n=0}^{N} \frac{\gamma_n}{n!} x^n$, with $\gamma_0 = 1$, $\gamma_n > 0$ for $n = 1, 2, \ldots$, and suppose that the Turán inequalities, $\gamma_n^2 \geq \alpha^2 \gamma_{n-1}\gamma_{n+1}$, hold for $n = 1, 2, \ldots, N-1$, where*

$$\alpha := \max\left(2, \frac{\sqrt{2}}{2}(1 + \sqrt{1+\gamma_1})\right). \tag{4.5}$$

Then the polynomial $\tilde{\varphi}(x) = \sum_{n=0}^{N} \gamma_n \binom{x}{n}$ has only real, simple negative zeros.

Corollary 4.10 ([24, Corollary 4.9]) *Let $\varphi(x) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} x^n$, with $\gamma_0 = 1$, $\gamma_n \geq 0$ for $n = 1, 2, 3, \ldots$, and suppose that*

$$\gamma_n^2 \geq \alpha^2 \gamma_{n-1}\gamma_{n+1}, \qquad \textit{where}$$

$$\alpha \geq \max\left(2, \frac{\sqrt{2}}{2}(1 + \sqrt{1+\gamma_1})\right). \tag{4.6}$$

Then $\varphi(x)$ and $\tilde{\varphi}(x) = \sum_{n=0}^{\infty} \gamma_n \binom{x}{n}$ are entire functions of order zero and φ, $\tilde{\varphi} \in \mathcal{L}\text{-}\mathcal{P}^+$.

In order to expedite our exposition, we shall also introduce the following definition.

Definition 4.11 A sequence $\{\gamma_k\}_{k=0}^{\infty}$ of nonnegative real numbers will be called a *rapidly decreasing sequence* if $\{\gamma_k\}_{k=0}^{\infty}$ satisfies inequality (4.6).

The sequence $\{e^{-ak^2}\}_{k=0}^{\infty}$ is rapidly decreasing if $a \geq \log 2$ and this sequence is a Laguerre sequence for any $a > 0$. Sequences of the form $\{e^{-ak^p}\}_{k=0}^{\infty}$, where $a > 0$ and p is a positive integer, $p \geq 3$, are multiplier sequences, but these sequences cannot be interpolated by functions $\varphi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$. For indeed, if $\varphi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$, then

$$\varphi(x) = e^{-\alpha x^2 + \beta x}\Pi(x) := e^{-\alpha x^2 + \beta x}\prod_{n=1}^{\infty}(1 + x/x_n)e^{-x/x_n}, \tag{4.7}$$

where $\alpha \geq 0,\ \beta \in \mathbb{R},\ x_n > 0$ and $\sum_{n=1}^{\infty} 1/x_n^2 < \infty$. Then from the standard estimates of the canonical product $\Pi(x)$ (see, for example, [12, p. 21]), we deduce that for any $\epsilon > 0$, there is a positive integer k_0 such that

$$\Pi(k) > e^{-k^{2+\epsilon}} \qquad (k \geq k_0). \tag{4.8}$$

We infer from (4.7) and (4.8) that complex zero decreasing sequences which decay at least as fast as $\{e^{-ak^3}\}_{k=0}^{\infty}$ cannot be interpolated by functions φ in $\mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$.

By way of applications of Corollary 4.10, we proceed to state two results which show how rapidly decreasing sequences can be used to generate complex zero decreasing sequences.

Corollary 4.12 ([24, Corollary 4.7]) *Let* $\{\gamma_k\}_{k=0}^{\infty}$*,* $\gamma_0 = 1$*,* $\gamma_k > 0$*, be a rapidly decreasing sequence. Then for each fixed* $t \geq \gamma_1$*,*

$$\tilde{\varphi}_t(x) = \sum_{j=0}^{\infty} \frac{\gamma_j}{t^j} \binom{x}{j} \in \mathcal{L}\text{-}\mathcal{P}^+.$$

Moreover, if $T_t = \{g_k(1/t)\}_{k=0}^{\infty}$*, where* $g_k(t) = \sum_{j=0}^{k} \binom{k}{j} \gamma_j t^j$ *is the kth Jensen polynomial associated with the sequence* $\{\gamma_k\}_{k=0}^{\infty}$*, then* T_t *is a CZDS for* $t \geq \gamma_1$*; that is, for any polynomial* $f(x) = \sum_{k=0}^{N} a_k x^k \in \mathbb{R}[x]$*, we have* $Z_c(T_t[f(x)]) \leq Z_c(f)$ *for* $t \geq \gamma_1$*, where* $T_t[f(x)] = \sum_0^N a_k g_k(1/t) x^k$*.*

Corollary 4.13 ([24, Corollary 4.8]) *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a rapidly decreasing sequence and let*

$$\beta_k = \sum_{j=0}^{k} \binom{k}{j} \gamma_j \tag{4.9}$$

Then the sequence $\{\beta_k\}_{k=0}^{\infty}$ *is a CZDS.*

We remark that if $\{\gamma_0, \gamma_1, \ldots, \gamma_n, 0, 0, \ldots\}$ is a CZDS with $\gamma_k > 0$ for $0 \leq k \leq n$, then the sequence $\{g_k(t)\}_{k=0}^{\infty}$, where $g_k(t) = \sum_{j=0}^{k} \binom{k}{j} \gamma_j t^j$, may not be a CZDS for some $t > 0$. To verify this claim, consider the sequence $T = \{1, 1, \frac{1}{2}, 0, 0, \ldots\}$. Then it follows that T is a CZDS [24, Proposition 3.5]. A calculation shows that $g_k(t) = 1 + kt + \frac{k(k-1)}{4} t^2$. Let $h_t(x) = 1 + xt + \frac{x(x-1)}{4} t^2$, so that $h_t(k) = g_k(t)$. But $h_t(x)$ has real zeros (both of which are positive) if and only if $t \geq 8$. Hence by Theorem 4.7, $\{g_k(t)\}_{k=0}^{\infty}$ is not a CZDS for any $t > 0$.

In contrast to the previous examples, it is possible to exhibit a CZDS $\{\gamma_k\}_{k=0}^{\infty}$ for which the sequence $\{g_k(t)\}_{k=0}^{\infty}$ is a CZDS *for all* $t > 0$, where $g_k(t) = \sum_{j=0}^{k} \binom{k}{j} \gamma_j t^j$. Let $\gamma_k = 1/k!$, $k = 0, 1, 2, \ldots$. Then $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS and for each fixed $t > 0$, $\{g_k(t)\}_{k=0}^{\infty}$ is a CZDS (cf. [24, Lemma 5.3]).

The principal source of the difficulty in characterizing CZDS is that, today, the only known, essentially nontrivial CZDS are the multiplier sequences that can be interpolated by functions in $\mathcal{L}\text{-}\mathcal{P}$. We use the terms "essentially nontrivial" advisedly to circumvent trivial examples of the following sort. Let $f(x) := 2 - \sin(\pi x)$. Then, the sequence $\{2, 2, 2, \ldots\}$ is clearly a CZDS, but $f(x) \notin \mathcal{L}\text{-}\mathcal{P}$. More sophisticated examples fostered a renewed scrutiny of the Karlin-Laguerre problem, and the investigation of when a CZDS can be interpolated by functions in $\mathcal{L}\text{-}\mathcal{P}^+$ has led to the following two theorems ([8], [9], [10]).

Theorem 4.14 ([9, Theorem 2]) *Let* $\{\gamma_k\}_{k=0}^{\infty}$*,* $\gamma_k > 0$*, be a CZDS. If*

$$\overline{\lim_{k \to \infty}} \gamma_k^{1/k} > 0, \tag{4.10}$$

then there is a function $\varphi(z) \in \mathcal{L}\text{-}\mathcal{P}$ *of the form*

$$\varphi(z) := be^{az}\psi(z) := be^{az}\prod_{n=1}^{\infty}\left(1+\frac{z}{x_n}\right),$$

where $a, b \in \mathbb{R}$, $b \neq 0$, $x_n > 0$ *and* $\sum_{n=1}^{\infty} 1/x_n < \infty$, *such that* $\varphi(z)$ *interpolates the sequence* $\{\gamma_k\}_{k=0}^{\infty}$*; that is,* $\gamma_k = \varphi(k)$ *for* $k = 0, 1, 2, \ldots$.

Theorem 4.15 ([8, Theorem 3.6]) *Let* $f(z)$ *be an entire function of exponential type. Suppose that* $\{f(k)\}_{k=0}^{\infty}$ *is a CZDS, where* $f(0) = 1$*. Let* $h_f(\theta)$ *denote the (Phragmén–Lindelöf) indicator function of* $f(z)$*, that is,*

$$h_f(\theta) := h(\theta) := \overline{\lim_{r\to\infty}}\,\frac{\log|f(re^{i\theta})|}{r}, \tag{4.11}$$

where $\theta \in [-\pi, \pi]$*. If* $h_f(\pm\pi/2) < \pi$*, then* $f(z)$ *is in* $\mathcal{L}\text{-}\mathcal{P}$ *and* $f(z)$ *can be expressed in the form*

$$f(z) = e^{az}\prod_{n=1}^{\infty}\left(1+\frac{z}{x_n}\right),$$

where $a \in \mathbb{R}$, $x_n > 0$ *and* $\sum_{n=1}^{\infty} 1/x_n < \infty$.

These theorems are complementary results in the following sense. Theorem 4.14 asserts that if a CZDS (of positive terms) does *not decay too fast* (cf. (4.10)), then the sequence can be interpolated by function in $\mathcal{L}\text{-}\mathcal{P}$ having only real negative zeros. In contrast, Theorem 4.15 says that if for some entire function, f, of exponential type, the sequence $\{f(k)\}_{k=0}^{\infty}$ is a CZDS and if f does *not grow too fast* along the imaginary axis (cf. (4.11)), then f has only real negative zeros. If a multiplier sequence does decay rapidly (cf. (4.12)), then the question whether or not such a sequence can be a CZDS remains an open problem.

Problem 4.16 If $\varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}^+$ (so that $\gamma_k \geq 0$) and if

$$\overline{\lim_{k\to\infty}}\,\gamma_k^{1/k} = 0, \tag{4.12}$$

then is $\{\gamma_k\}_{k=0}^{\infty}$ is a CZDS?

The proof of Theorem 4.14 is rather involved and technical and therefore, due to restrictions of space, it would be difficult to convey here the flavor of the arguments used in [9]. By confining our attention to some special cases of Theorem 4.14, we propose to sketch here some of the techniques and results that can be used to establish converses of Laguerre's theorem (Theorem 4.1). In the case of polynomials, the converse of Laguerre's theorem is an immediate consequence of Theorem 4.7 since this theorem completely characterizes the class of all polynomials which interpolate CZDS. On the other hand, the converse of Laguerre's theorem fails, in general, for transcendental entire functions, as the following example shows.

Example 4.17 Let $p(x)$ be a polynomial in $\mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$ (so that the sequence $\{p(k)\}_{k=0}^{\infty}$ is a CZDS). Then, as noted earlier,

$$\varphi_1(x) := \frac{1}{\Gamma(-x)} + p(x) \qquad \text{and} \qquad \varphi_2(x) := \sin(\pi x) + p(x)$$

are transcendental entire functions which both interpolate the same sequence $\{p(k)\}_{k=0}^{\infty}$, but these entire functions are *not* in $\mathcal{L}\text{-}\mathcal{P}$. Thus, in the transcendental case additional hypotheses are required in order that the converse of Laguerre's theorem hold.

The main result in [26, Theorem 3.9] shows that the converse of Laguerre's theorem is valid for (transcendental) entire functions of the form $\varphi(x)p(x)$, where $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ and $p(x)$ is a real polynomial which has no nonreal zeros in the left half-plane. The proof hinges on a deep result of Schoenberg (see Theorem 4.19 below) on the representation of the reciprocals of functions $\varphi \in \mathcal{L}\text{-}\mathcal{P}I$ in terms of Pólya frequency functions. These functions are defined as follows.

Definition 4.18 A function $K : \mathbb{R} \to \mathbb{R}$ is a *frequency function* if it is a nonnegative measurable function such that

$$0 < \int_{-\infty}^{\infty} K(s)\, ds < \infty.$$

A frequency function K is said to be a *Pólya frequency function* if it satisfies the following condition: For every two sets of increasing real numbers $s_1 < s_2 < \cdots < s_n$ and $t_1 < t_2 < \cdots < t_n$ $(n = 1, 2, 3, \ldots)$, the determinantal inequality

$$\begin{vmatrix} K(s_1 - t_1) & K(s_1 - t_2) & \ldots & K(s_1 - t_n) \\ K(s_2 - t_1) & K(s_2 - t_2) & \ldots & K(s_2 - t_n) \\ & & \ldots & \\ K(s_n - t_1) & K(s_n - t_2) & \ldots & K(s_n - t_n) \end{vmatrix} \geq 0$$

holds.

Theorem 4.19 (Schoenberg [79, p. 354]) *Suppose that $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}I$ where $\varphi(x) > 0$ if $x > 0$ and $\varphi(x)$ is not of the form $ce^{\beta x}$. Then the reciprocal of φ can be represented in the form*

$$\frac{1}{\varphi(z)} = \int_0^\infty e^{-sz} K(s)\, ds, \qquad \operatorname{Re} z > 0,$$

where $K(s)$ is a Pólya frequency function such that $K(s) = 0$ if $s < 0$ and the integral converges up to the first pole of $\dfrac{1}{\varphi(z)}$. Conversely, suppose that $K(s)$ is a Pólya frequency function such that $K(s) = 0$ for $s < 0$ and the integral converges for $\operatorname{Re} z > 0$. Then this integral represents, in the half-plane $\operatorname{Re} z > 0$, the reciprocal of a function $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}I$, where $\varphi(x)$ is not of the form $ce^{\beta x}$.

Theorem 4.20 ([26, Theorem 3.5]) *Let $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, where $\varphi(x)$ is not of the form $ce^{\beta x}$, $c, \beta \in \mathbb{R}$. Let $p(x)$ be a polynomial with only real zeros, and suppose that $\varphi(0)p(0) = 1$. Then the sequence $T = \{\varphi(k)p(k)\}_{k=0}^{\infty}$ is a CZDS if and only if p has only real negative zeros.*

If $p(x)$ has only real negative zeros, then $\varphi(x)p(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ and T is a CZDS, by Laguerre's theorem. Conversely, suppose that T is a CZDS. With *reductio ad absurdum* in mind, assume that $p(x)$ has a positive zero. Since T is a CZDS, the sequence $\{\frac{1}{\varphi(k)p(k)}\}_{k=0}^{\infty}$ is a λ-sequence and so the application of this sequence to the positive function e^{-x} must give (see the remarks after Definition 4.3)

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!\varphi(k)p(k)} \geq 0$$

for all $x \in \mathbb{R}$. Since $\varphi(x)$ is not of the form $ce^{\beta x}$, we may invoke Schoenberg's theorem (Theorem 4.19) and therefore we can express $F(x)$ as

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! p(k)} \int_0^\infty K(s) e^{-ks}\, ds,$$

where $K(s)$ is a Pólya frequency function such that $K(s) = 0$ for $s < 0$. Now a somewhat complicated analysis of the behavior of $F(x)$ shows that $F(x) \to -\infty$ as $x \to \infty$. Consequently, $\{\frac{1}{\varphi(k)p(k)}\}_{k=0}^{\infty}$ is not a λ-sequence and so we have obtained the desired contradiction.

The next preparatory result, whose proof also depends on Schoenberg's theorem, provides information about the oscillation properties of entire functions under the action of certain λ-sequences.

Proposition 4.21 ([26, Proposition 3.7]) *Let $a < 0$, $b \in \mathbb{R}$ and $4b - a^2 > 0$. Suppose that $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$ with $\varphi(x) > 0$ if $x \geq 0$ and φ is not of the form $ce^{\beta x}$. Then the function*

$$F(x,a,b) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!(k^2+ak+b)\varphi(k)} \tag{4.13}$$

changes sign infinitely often in the interval $(0, \infty)$.

With the aid of the foregoing preliminary results, we proceed to prove the following theorem.

Theorem 4.22 ([26, Theorem 3.8]) *Suppose that $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, where $\varphi(0) > 0$ and $\varphi(x)$ is not of the form $ce^{\beta x}$. Let $p(x)$ be a real polynomial all of whose zeros lie in the right half-plane $\operatorname{Re} z > 0$. Let $h(x) = p(x)\varphi(x)$. If the sequence $T = \{h(k)\}_{k=0}^{\infty}$ is a CZDS, then all the zeros of $p(x)$ are real.*

Proof. Assume the contrary so that $h(x)$ may be expressed in the form $h(x) = \tilde{g}(x)(x^2+ax+b)\varphi(x)$, where $x^2+ax+b = (x+\alpha)(x+\bar{\alpha})$ and $\alpha = \frac{a}{2} + i\tau$, $\tau = \frac{\sqrt{4b-a^2}}{2}$, $4b - a^2 > 0$ and $\operatorname{Re}\alpha = \frac{a}{2} < 0$. Then the polynomial $\tilde{g}(x)$ gives rise to the entire function $\sum_{k=0}^{\infty} \frac{\tilde{g}(k)(-1)^k x^k}{k!} = g(x)e^{-x}$, where $g(x)$ is a polynomial. We next approximate the entire function $g(x)e^{-x}$ by means of the polynomials $q_n(x) = g(x)\left[\left(1 - \frac{x}{2n}\right)^{2n} + \epsilon_n\right]$, where $\epsilon_n > 0$ and $\lim_{n\to\infty} \epsilon_n = 0$ (see the remarks following Definition 4.3). We note, in particular, that $q_n(x)$ has exactly the same real zeros as $g(x)$ has. Moreover, as $n \to \infty$, $q_n(x) \to g(x)e^{-x}$ uniformly on compact subsets of $\mathbb{C}$. If we set $\Lambda = \{\frac{1}{h(k)}\}_{k=0}^{\infty}$, then by Proposition 4.21, the function

$$\Lambda[g(x)e^{-x}] = F(x,a,b) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!(k^2+ak+b)\varphi(k)}$$

has infinitely many sign changes in the interval $(0, \infty)$. Also, as $n \to \infty$, $f_n(x) := \Lambda[q_n(x)]$ converges to $F(x,a,b)$ uniformly on compact subsets of $\mathbb{C}$. Thus, for all sufficiently large n, each of the approximating polynomials $f_n(x)$ has more real zeros than $g(x)$ has. Since T is a CZDS, $Z_c([T[f_n(x)]) \leq Z_c(f_n(x))$, and since $\deg q_n = \deg f_n$ consequently, for all n sufficiently large, the polynomial $T[f_n(x)] = T[\Lambda[q_n(x)]] = q_n(x)$ has more real zeros than $g(x)$ has. This is the desired contradiction. □

Combining Theorem 4.22 with Theorem 4.20 (for the details see [26, Theorem 3.9 and Proposition 3.1]) yields the following converse of Laguerre's theorem.

Theorem 4.23 ([26, Theorem 3.9]) *Suppose that $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^+$, where $\varphi(0) > 0$. Let $p(x)$ be a real polynomial with no nonreal zeros in the left half-plane $\operatorname{Re} z < 0$. Suppose that $p(0)\varphi(0) = 1$ and set $h(x) = p(x)\varphi(x)$. Then $T = \{h(k)\}_{k=0}^{\infty}$ is a CZDS if and only if $p(x)$ has only real negative zeros.*

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