The Fox–Wright functions and Laguerre multiplier sequences

Thomas Craven *, George Csordas

Department of Mathematics, University of Hawaii, Honolulu, HI 96822, USA

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Abstract

Linear operators which (1) preserve the reality of zeros of polynomials having only real zeros and (2) map stable polynomials into stable polynomials are investigated using recently established results concerning the zeros of certain Fox–Wright functions and generalized Mittag-Leffler functions. The paper includes several open problems and questions.

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1. Introduction

The course of our investigation involving the distribution of zeros of entire functions, has led us to the following open problems.

Problem 1.1. Characterize the meromorphic functions $F(x) = \varphi(x)/\psi(x)$, where $\varphi$ and $\psi$ are entire functions such that the polynomial $\sum_{k=0}^{n} F(k) a_k x^k$ has only real zeros whenever the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros.

* Corresponding author.
E-mail addresses: tom@math.hawaii.edu (T. Craven), george@math.hawaii.edu (G. Csordas).
Problem 1.2. Characterize the meromorphic functions $F(x)$ with the property that
\[ \sum_{k=0}^{\infty} F(k) a_k x^k / k! \] is a transcendental entire function with only real zeros (or the zeros all lie in the half-plane $\text{Re} \, z < 0$) whenever the entire function $\sum_{k=0}^{\infty} a_k x^k / k!$ has only real zeros.

These problems appear to be new and are ostensibly difficult. While at the present time we are unable to solve them, we wish to call attention to them and to related questions and prove some theorems which establish the existence of some non-trivial meromorphic functions which possess the desired properties. In the special case when $F(x)$ is an entire function, open Problem 1.1 is intimately connected to a classical theorem of Laguerre (see Theorem 2.4). The specific entire functions we will consider in the sequel are members of the class of generalized Fox–Wright functions which recently have played an increasingly significant role in various types of applications (see [12,13,36,37]).

The Fox–Wright function (cf. [12,20]) is defined as
\[ p\Psi q(x) := \sum_{k=0}^{\infty} \prod_{j=1}^{p} \Gamma(a_j k + b_j) x^k \prod_{j=1}^{q} \Gamma(c_j k + d_j) k!, \] (1.1)
where $\Gamma(x)$ denotes the gamma function and $p$ and $q$ are nonnegative integers. If we set $b_j = 1$ ($j = 1, 2, 3, \ldots, p$) and $d_j = 1$ ($j = 1, 2, 3, \ldots, q$), then (1.1) reduces to the familiar generalized hypergeometric function (cf. [31])
\[ pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; x) := \sum_{k=0}^{\infty} (a_1)_k \cdots (a_p)_k x^k (b_1)_k \cdots (b_q)_k k!, \] (1.2)
where the Pochhammer or ascending factorial symbol for $a \in \mathbb{C} \setminus \{0\}$ is defined as $(a)_0 = 1$, $(a)_k = a(a+1)(a+2)\ldots(a+k-1) = \Gamma(a+k)/\Gamma(a)$, $k = 1, 2, 3, \ldots$. Henceforth, we will confine our attention to those Fox–Wright functions and generalized hypergeometric functions which are entire functions.

Today, notwithstanding the prolific research activity in this area, there is little known about the distribution of zeros of the Fox–Wright function and the generalized hypergeometric function, except in some very special cases. Consider, for example, the entire functions
\[ 1\Psi_1(x) = \sum_{k=0}^{\infty} \frac{\Gamma(ak + 1) x^k}{\Gamma(ck + 1) k!}, \quad c \geq a \geq 0, \quad \text{and} \] (1.3)
\[ 2\Psi_1(x) = \sum_{k=0}^{\infty} \frac{\Gamma(ak + 1)\Gamma(bk + 1) x^k}{\Gamma(ck + d + 1) k!}, \quad a, b, c, d \geq 0 \quad \text{and} \quad c \geq a + b. \] (1.4)

With the parameter values $a = 1$ and $c = \alpha > 0$ in (1.3) (here we dispense with the requirement that $c \geq a$), $1\Psi_1(x)$ is just the classical Mittag-Leffler function (see, for example, [11, vol. 3, Chapter XVIII])
\[ E_{\alpha}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \] (1.5)
$E_\alpha(x)$ is a generalization of the exponential function and furnishes important examples of entire functions of finite order. If $\alpha \geq 2$, then it is known [25, Corollary 3] that the entire function $E_\alpha(x)$ (of order $1/\alpha$) has only real negative zeros. Now the generalized Mittag-Leffler function

$$E_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,$$

is a special case of $2\Psi_1(x)$ with $a = 0, b = 1, c = \alpha$ and $d + 1 = \beta$. When $\beta = 1$, $E_\alpha(x) = E_{\alpha,1}(x)$ and so $E_1(x) = e^x$.

**Remark 1.3.** The generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ plays an important role in analysis where it is used in the theory of integral transforms and representations of complex-valued functions [8–10], fractional calculus [19], and other areas [20,25,29,30, 33–35,39–41].

This paper is organized as follows. In Section 2, we briefly review a few definitions and results involving multiplier sequences. We provide a number of examples of Laguerre multiplier sequences (Examples 2.6 and 2.7) and with the aid of some recent results about the zeros of the generalized Mittag-Leffler function, we establish the existence of multiplier sequences which satisfy the properties stipulated in Problem 1.1 (Example 2.8 and Theorem 2.11). In Section 3, we use stability analysis (Lemma 3.5), in conjunction with the Hermite–Biehler theorem (Theorem 3.1) to prove that certain Mittag-Leffler-type functions have only real zeros (Theorem 3.6 and Corollary 3.7). We also prove that the sequence $\{(2k)!/(3k)!\}_{k=0}^{\infty}$ is a multiplier sequence (Theorem 3.13) and, as a consequence of this result, we are able to show that a class of entire Fox–Wright functions possess only real negative zeros (Theorem 3.14 and Corollary 3.16). The paper includes several open problems, conjectures and questions related to Problems 1.1 and 1.2, introduced above.

## 2. Multiplier sequences and the Mittag-Leffler function

Since Problems 1.1 and 1.2 involve a class of linear operators, known as multiplier sequences (see Definition 2.2), we first recall briefly a few definitions and pertinent results which will facilitate our investigation of these problems.

To begin with, we associate with each sequence $\{\gamma_k\}_{k=0}^{\infty}$ of real numbers, a linear operator $L$ whose action on monomials is defined by $L[x^k] := \gamma_k x^k, k = 0, 1, 2, \ldots$. Thus, if $p(x) := \sum_{k=0}^{n} a_k x^k$, then

$$L[p(x)] = L\left[\sum_{k=0}^{n} a_k x^k\right] = \sum_{k=0}^{n} \gamma_k a_k x^k.$$  

In the classical setting, the problem (solved by Pólya and Schur [28]) is to characterize all real sequences $L = \{\gamma_k\}_{k=0}^{\infty}, \gamma_k \in \mathbb{R}$, such that if a polynomial $p(x)$ has only real zeros, then the polynomial $L[p(x)]$ also has only real zeros.
Definition 2.1. A real entire function \( \varphi(x) := \sum_{k=0}^{\infty} (\gamma_k/k!) x^k \) is said to be in the Laguerre–Pólya class, written \( \varphi(x) \in \mathcal{L}-\mathcal{P} \), if \( \varphi(x) \) can be expressed in the form

\[
\varphi(x) = c x^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k},
\]

where \( c, \beta, x_k \in \mathbb{R}, c \neq 0, \alpha \geq 0, n \) is a nonnegative integer and \( \sum_{k=1}^{\infty} 1/x_k^2 < \infty \). If \( -\infty \leq a < b \leq \infty \) and if \( \varphi(x) \in \mathcal{L}-\mathcal{P} \) has all its zeros in \( (a, b) \) (or \( [a, b] \)), then we will use the notation \( \varphi \in \mathcal{L}-\mathcal{P}(a, b) \) (or \( \varphi \in \mathcal{L}-\mathcal{P}[a, b] \)). If \( \gamma_k \geq 0 \) (or \( (-1)^k \gamma_k \geq 0 \) or \( -\gamma_k \geq 0 \)) for all \( k = 0, 1, 2, \ldots \), then \( \varphi \in \mathcal{L}-\mathcal{P} \) is said to be of type I in the Laguerre–Pólya class, and we will write \( \varphi \in \mathcal{L}-\mathcal{P}^I \). If \( \varphi \in \mathcal{L}-\mathcal{P} \ \setminus \mathcal{L}-\mathcal{P}^I \), then \( \varphi \) is said to be of type II in the Laguerre–Pólya class, and we will write \( \varphi \in \mathcal{L}-\mathcal{P}^II \). We will also write \( \varphi \in \mathcal{L}-\mathcal{P}^+ \), if \( \varphi \in \mathcal{L}-\mathcal{P}^I \) and \( \gamma_k \geq 0 \) for all \( k = 0, 1, 2, \ldots \).

Definition 2.2. A sequence \( L = \{\gamma_k\}_{k=0}^{\infty} \) of real numbers is called a multiplier sequence of the first kind or just a multiplier sequence if, whenever the real polynomial \( p(x) = \sum_{k=0}^{n} a_k x^k \) has only real zeros (i.e., \( p \in \mathcal{L}-\mathcal{P} \)) the polynomial \( L[p(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k \) also has only real zeros. A sequence \( L = \{\gamma_k\}_{k=0}^{\infty} \) of real numbers is called a multiplier sequence of the second kind, if whenever \( p(x) \) has only real negative zeros, the polynomial \( L[p(x)] \) has only real zeros.

The following are well-known characterizations of multiplier sequences (cf. [28,38], [27, pp. 100–124] or [24, pp. 29–47]). A sequence \( L = \{\gamma_k\}_{k=0}^{\infty} \) is a multiplier sequence if and only if

\[
\varphi(x) = L[e^x] := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}-\mathcal{P}^I. \tag{2.3}
\]

Similarly, \( L = \{\gamma_k\}_{k=0}^{\infty} \) is a multiplier sequence of the second kind if and only if

\[
\varphi(x) = L[e^x] := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}-\mathcal{P}^II. \tag{2.4}
\]

Moreover, the algebraic characterization of multiplier sequences asserts that a sequence \( L = \{\gamma_k\}_{k=0}^{\infty} \) is a multiplier sequence if and only if

\[
g_n(x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j \in \mathcal{L}-\mathcal{P}^I \quad \text{for all } n = 1, 2, 3, \ldots \tag{2.5}
\]

The polynomials \( g_n(x) \) are called the Jensen polynomials associated with the entire function \( \varphi(x) \) defined by (2.3). We also mention that the Jensen polynomials (2.5) are the “canonical” approximating polynomials associated with the entire function defined by (2.4). Indeed, for nonnegative integers \( m \) and \( n \) let

\[
g_{n,m}(z) := \sum_{k=0}^{n} \binom{n}{k} \gamma_{k+m} z^k. \tag{2.6}
\]
Then it is known, and in any case it is easy to show, that for 
\(m = 0, 1, 2, \ldots\),
\[
\lim_{n \to \infty} g_{n,m} \left( \frac{z}{n} \right) = \varphi^{(m)}(z),
\] (2.7)
uniformly on compact subsets of \(\mathbb{C}\).

In the statement of an extended version of Laguerre’s theorem (cf. Theorem 2.4) it will be convenient to adopt the following terminology (cf. [4]).

**Definition 2.3.** We say that a sequence \(\{\gamma_k\}_{k=0}^\infty\) is a complex zero-decreasing sequence (CZDS), if
\[
Z_c \left( \sum_{k=0}^n \gamma_k a_k x^k \right) \leq Z_c \left( \sum_{k=0}^n a_k x^k \right),
\] (2.8)
for any real polynomial \(\sum_{k=0}^n a_k x^k\), where \(Z_c(p)\) denotes the number of non-real zeros of the polynomial \(p\), counting multiplicities. (The acronym CZDS will also be used in the plural.)

**Theorem 2.4** (Laguerre [24, Satz 3.2], [5,26]).

1. Let \(f(x) = \sum_{k=0}^n a_k x^k\) be an arbitrary real polynomial of degree \(n\) and let \(h(x)\) be a polynomial with only real zeros, none of which lie in the interval \((0, n)\). Then
   \[Z_c(\sum_{k=0}^n h(k) a_k x^k) \leq Z_c(f(x)).\]
2. Let \(f(x) = \sum_{k=0}^n a_k x^k\) be an arbitrary real polynomial of degree \(n\), let \(\varphi \in \mathcal{L}-\mathcal{P}\) and suppose that none of the zeros of \(\varphi\) lie in the interval \((0, n)\). Then the inequality
   \[Z_c(\sum_{k=0}^n \varphi(k) a_k x^k) \leq Z_c(f(x))\]
holds.
3. Let \(\varphi \in \mathcal{L}-\mathcal{P}(-\infty, 0]\), then the sequence \(\{\varphi(k)\}_{k=0}^\infty\) is a complex zero decreasing sequence.

In the sequel it will be convenient to adopt the following terminology. We will term a sequence \(\{\varphi(k)\}_{k=0}^\infty\) a Laguerre multiplier sequence, if \(\varphi \in \mathcal{L}-\mathcal{P}(-\infty, 0]\). (There are, of course, multiplier sequences, as for example, \(\{1 + k + k^2\}_{k=0}^\infty\), which do not arise in this manner.) A multiplier sequence of the form \(\{F(k)\}_{k=0}^\infty\), where \(F(x)\) is a meromorphic function, will be termed a meromorphic Laguerre multiplier sequence. The existence of the latter sequences will be established below for functions with poles. Those without poles are handled by Theorem 2.4.

**Problem 2.5.** Suppose that \(\{F(k)\}_{k=0}^\infty\) is a meromorphic Laguerre multiplier sequence. Is the sequence \(\{F(k)\}_{k=0}^\infty\) a CZDS?

**Example 2.6** (Pólya [27, p. 234]). For \(\alpha = 2, 3, 4, \ldots\), the classical Mittag-Leffler function \(E_\alpha(x) = \sum_{k=0}^\infty x^k / \Gamma(\alpha k + 1) \in \mathcal{L}-\mathcal{P}^+\). Consider \(\varphi(x) := \Gamma(x + 1) / \Gamma(\alpha x + 1)\) for a fixed, positive even integer \(\alpha\). Observe that the poles of \(\Gamma(x + 1)\) are canceled by the zeros of \(1 / \Gamma(\alpha x + 1)\) and whence by Theorem 2.4 the sequence \(L = \{\varphi(k)\}_{k=0}^\infty\) is a CZDS and so a multiplier sequence. Thus, using the results about the Jensen polynomials (cf. (2.3),
(2.5) and (2.7)), we conclude that $L[e^x] = E_\alpha(x) \in \mathcal{L}_{-\mathcal{P}}^+$. (When $\alpha$ is not an integer see the comment in [25, p. 284] and Example 2.8.)

**Example 2.7.** Since for any $a, b > 0$, the entire function $1/\Gamma(ax + b) \in \mathcal{L}_{-\mathcal{P}}(-\infty, 0)$, it follows from Theorem 2.4 that the Fox–Wright function $\sum_{k=0}^{\infty}(1/\Gamma(ak + 1))(x^k/k!) \in \mathcal{L}_{-\mathcal{P}}^+$. By mimicking the argument used in Example 2.6, we see that the more general Fox–Wright function

$$p\Psi_q(x) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j k + 1)}{\prod_{j=1}^{q} \Gamma(2a_j k + 2)} \frac{x^k}{k!} \in \mathcal{L}_{-\mathcal{P}}^+,$$

for $p \leq q$ and $a_j > 0$ ($j = 1, 2, 3, \ldots$).

In the sequel we will use properties of the Mittag-Leffler function to produce examples of meromorphic Laguerre multiplier sequences; that is, sequences which satisfy the assumptions stipulated in Problem 1.1.

**Example 2.8.** Let

$$L_{\alpha, \beta} := \left\{ \frac{\Gamma(k + 1)}{\Gamma(\alpha k + \beta)} \right\}_{k=0}^{\infty}, \quad (2.9)$$

Then for $1 \leq \beta < 3$, $L_{2, \beta}$ is a meromorphic Laguerre multiplier sequence. (This follows from the results in [10] and [9, Theorem 1.1].) These authors have shown that $E_{2, \beta}(x) \in \mathcal{L}_{-\mathcal{P}}^+$ when $1 \leq \beta < 3$. Recently it has been proved that for all $\alpha \geq 2$, $E_{\alpha, \beta}(x) \in \mathcal{L}_{-\mathcal{P}}^+$ when $\beta = 1$ or $\beta = 2$ [25, Corollary 3]. Thus, for $\alpha \geq 2$ the sequences $L_{\alpha, 1}$ and $L_{\alpha, 2}$ are meromorphic Laguerre multiplier sequences.

**Problem 2.9.** Are the sequences $L_{\alpha, 1}$ and $L_{\alpha, 2}$, $\alpha \geq 2$, CZDS?

**Remark 2.10.** We remark that there are other non-trivial examples of meromorphic Laguerre multiplier sequences which come to light from some deep investigations of the distribution of zeros of the generalized Mittag-Leffler function. A.Y. Popov has shown that for every integer $n$, $n \geq 3$, the zeros of $E_{n, n+1}$ lie in the interval $(-\infty, -(2n)!/n!)$ (see [29]). A comprehensive analysis of the asymptotic behavior of the zeros of the generalized Mittag-Leffler function was given by A.M. Sedletskii in a series of papers [33–35]. In particular, he has shown ([33]; see also [34, Theorem A] and [29, p. 643]) that if $\{z_n\}_{n=1}^{\infty}$ denotes the zeros of $E_{\alpha, \beta}$, arranged in ascending order of absolute values, then asymptotically we have

$$z_n = -\left(\frac{\pi}{\sin(\pi/\alpha)}\left(n + \frac{1}{2} + \frac{\beta - 1}{\alpha} + r_n\right)\right)^{\frac{1}{\alpha}}, \quad n \to \infty, \quad (2.10)$$

for $\alpha > 2$, $\beta \in \mathbb{C}$, where $|r_n|$ decreases exponentially as $n \to \infty$.

We will use (2.10) to prove the following theorem.
Theorem 2.11. Let $\alpha > 2$ and $\beta > 0$. Then there exists a positive integer $m_0 := m_0(\alpha, \beta)$ such that

$$E^{(m)}_{\alpha, \beta}(x) = \sum_{k=0}^{\infty} \frac{\Gamma(m + k + 1)}{\Gamma(\alpha (k + m) + \beta)} \frac{x^k}{k!} \in \mathcal{L} - \mathcal{P}^+, \quad m \geq m_0.$$  \hfill (2.11)

Thus, the sequence $\{\Gamma(m + k + 1)/\Gamma(\alpha (k + m) + \beta)\}_{k=0}^{\infty}$ is a meromorphic Laguerre multiplier sequence, for all nonnegative integers $m$ sufficiently large.

Proof. For fixed $\alpha > 2$ and $\beta > 0$, we infer from the asymptotic statement (2.10) that for all $n$ sufficiently large, the zeros of the entire function, $E_{\alpha, \beta}(x)$ are real, negative and simple. Thus, the real entire function $E_{\alpha, \beta}(x)$ (of order $1/\alpha < 2$) has at most a finite number of nonreal zeros. Now we recall that the celebrated Pólya–Wiman theorem \cite{3,6,17,18} states that if $f(x) = \phi(x)p(x)$, where $\phi(x) \in \mathcal{L} - \mathcal{P}$ and $p(x)$ is a real polynomial, then the successive derivatives of $f(x)$, from a certain one onward, have only real zeros. Hence, it follows that there exists a positive integer $m_0$ such that $E^{(m)}_{\alpha, \beta}(x) \in \mathcal{L} - \mathcal{P}^+$ for all $m \geq m_0$. Since the second assertion of the theorem is clear, the proof of the theorem is complete. \hfill \Box

3. Stability, multiplier sequences and the zeros of certain Fox–Wright functions

A real polynomial, $p(x)$, is said to be a stable polynomial (or a Hurwitz polynomial), if all the zeros of $p(x)$ lie in the open left half-plane, Re $z < 0$. The celebrated Routh–Hurwitz theorem provides a necessary and sufficient condition for a polynomial to be stable (see, for example, \cite[§40]{22}, \cite[§23]{24}). The importance of stability in analysis and matrix theory are well known (cf. the references in \cite{16}). For an elementary derivation of the three basic results in the Routh–Hurwitz theory; namely, the Hermite–Biehler theorem, the Routh–Hurwitz criterion and the total positivity of a Hurwitz matrix, we refer to \cite{16}. However, it seems that none of these familiar results, with the notable exception of the Hermite–Biehler theorem (Theorem 3.1) provide a tractable stratagem for proving our results in this section.

In the proof of Theorem 3.6, we will appeal to the following classical version of the Hermite–Biehler theorem.

Theorem 3.1 (The Hermite–Biehler theorem ([21, p. 305], [24, p. 13], [16])). Let

$$f(z) := p(z) + iq(z) =: a_n \prod_{k=1}^{n} (z - z_k), \quad 0 \neq a_n \in \mathbb{R},$$

where $p(z)$ and $q(z)$ are real polynomials of degree $\geq 2$. If function $f(z)$ has all its zeros in Im $z > 0$, then $p$ and $q$ have only real, simple zeros which interlace (separate one another) and $d(x) := q'(x)p(x) - q(x)p'(x) > 0$ for all real $x$.

Extensions of the Hermite–Biehler theorem to transcendental entire functions can be found in \cite[Chapter VII]{21}. In stability analysis it is frequently advantageous to consider operations which preserve stability. We will now mention examples of such operations. The Hadamard product (or
composition), \((p * q)(x)\), of two polynomials \(p(x) = \sum_{k=0}^{n} a_k x^k\) and \(q(x) = \sum_{k=0}^{m} b_k x^k\) is defined by \((p * q)(x) := \sum_{k=0}^{v} a_k b_k x^k\), where \(v = \min\{n, m\}\).

**Theorem 3.2** (Garloff and Wagner [14]). *The Hadamard product of two stable polynomials is a stable polynomial.*

Observe that a special case of this result is the Malo–Schur–Szegö composition theorem [5, Theorem 2.4]. The next result, which can be proved with the aid of the Hermite–Biehler theorem, states that nonnegative multiplier sequences preserve stability.

**Theorem 3.3** ([2, p. 421] or [21, p. 343]). *Let \(L\) be a nonnegative multiplier sequence. Then \(L[p(z)]\) is a stable polynomial, whenever \(p(z)\) is a stable polynomial.*

**Remark 3.4.** By the classical Gauss–Lucas theorem [22, p. 22], the derivative of a stable polynomial is a stable polynomial. This is also clear by Theorem 3.3. Indeed, one of the simplest (nonnegative) multiplier sequences is the sequence \([0, 1, 2, \ldots]\) which corresponds to the differentiation operator (after clearing the factor \(z\)). We also remark that increasing, nonnegative multiplier sequences can be completely characterized in terms of a Gauss–Lucas-type property [2, Theorem 2.8].

Preliminaries aside, we now proceed to prove the following stability result.

**Lemma 3.5.** *The Jensen polynomial*

\[
g_n(x) := \sum_{k=0}^{n} \binom{n}{k} k! \frac{k!}{(2k)!} x^k,
\]

*associated with the entire function*

\[
f(x) := \sum_{k=0}^{\infty} \frac{k!}{(2k)!} x^k,
\]

*is a stable polynomial.*

**Proof.** With the aid of the Gauss quadrature formula [7, Remark 3] one can show that the polynomial

\[
q_n(x) := \sum_{k=0}^{n} T_n^{(k)}(1)x^k,
\]

where \(T_n(x)\) denotes the \(n\)th Chebyshev polynomial of the first kind [32], is a stable polynomial [7, Corollary 1]. In terms of powers of \(x\), \(T_n(x)\), can be written explicitly ([23, p. 24] or [32, p. 37]) as

\[
T_n(x) = \frac{n}{2} \sum_{k=0}^{[n/2]} \frac{(-1)^k(n-k-1)!}{(n-2k)!k!} (2x)^{n-2k},
\]
where \([x]\) denotes the greatest integer less than or equal to \(x\). Now a calculation, together with an induction argument, shows (cf. [32, p. 38, Exercises 1.5.5 and 1.5.6]) that for \(0 \leq k \leq \lfloor n/2 \rfloor\),

\[
T_n^{(k)}(1) = n \frac{(n + k - 1)!2^k(k)!}{(n - k)!(2k)!}.
\] (3.2)

Substituting (3.2) in (3.1), we find that the polynomial

\[
Q_n(x) := \frac{1}{n}q_n\left(\frac{x}{2}\right) = \sum_{k=0}^{n} \frac{(n + k - 1)!}{(n - k)!(2k)!} x^k
\] (3.3)

is a stable polynomial. Since for each positive integer \(n\), \(L = \{1/\Gamma(n + k)\}_{k=0}^{\infty}\) is a (Laguerre) multiplier sequence, it follows from Theorem 3.3 that the polynomial \(L[Q_n(x)] = g_n(x)\) is a stable polynomial. \(\square\)

**Theorem 3.6.** The entire function

\[
\varphi(x) := F_2\left(1; \frac{1}{4}, \frac{3}{4}; \frac{x}{64}\right) = \sum_{k=0}^{\infty} \frac{(2k)!}{(4k)!} x^k \in \mathcal{L}-\mathcal{P}^+.
\] (3.4)

Moreover,

\[
L_1 = \left\{ \frac{(\Gamma(2k+1)\Gamma(k+1))}{\Gamma(4k+1)} \right\}_{k=0}^{\infty}
\]

is a Laguerre multiplier sequence.

**Proof.** By Lemma 3.5, we know that the Jensen polynomial

\[
g_n(x) := \sum_{k=0}^{n} \binom{n}{k} \frac{k!k!}{(2k)!} x^k
\]

is a stable polynomial; that is, the zeros of \(g_n(x)\) lie in the open left half-plane, \(\text{Re} \ z < 0\). Hence, via the clockwise rotation \(x \rightarrow -ix\), the zeros of \(g_n(-ix)\) lie in the open upper half-plane, \(\text{Im} \ z > 0\). Therefore, it follows from the Hermite–Biehler theorem that the zeros of the polynomial

\[
h_n(x) := \text{Re} \ g_n(-ix) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2k)!}{(n - 2k)!(4k)!} x^{2k}
\]

are all real and negative; that is, \(h_n(x) \in \mathcal{L}-\mathcal{P}^+\). But then the polynomial

\[
p_n(x) := (2n)!h_{2n}\left(\sqrt{-x}\right) = \sum_{k=0}^{n} \frac{(2n)!(2k)!}{(2n - 2k)!(4k)!} x^k \in \mathcal{L}-\mathcal{P}^+,
\]

for \(n = 1, 2, 3, \ldots\). Thus,
\[ p_n \left( \frac{x}{2n} \right) = \sum_{k=0}^{n} \frac{(2n)(2n-1) \ldots (2n-k+1) (2k)! (2n^k)}{(2n)^k (4k)!} \left( \frac{x^k}{4^k} \right) \rightarrow \sum_{k=0}^{\infty} \frac{(2k)! x^k}{(4k)!} = \varphi(x) \quad \text{as } n \to \infty, \]

uniformly on compact subsets of \( \mathbb{C} \). Therefore, \( \varphi(x) \in \mathcal{L}-\mathcal{P}^+ \) and whence the second assertion of the theorem is clear. \( \Box \)

Now it follows from the asymptotic formulae (2.10) for the zeros of the generalized Mittag-Leffler function, \( E_{2,\beta}(x), \beta > 0 \), that all the zeros \( E_{2,1/2} \), sufficiently large in absolute value, are real and negative. As a consequence of Theorem 3.6, we obtain the stronger result that all the zeros of \( E_{2,1/2}(x) \) are real and negative.

**Corollary 3.7.** The Mittag-Leffler function \( E_{2,1/2}(x) \in \mathcal{L}-\mathcal{P}^+ \).

**Proof.** By the Legendre duplication formula [31, p. 23],

\[
\frac{1}{\Gamma(2k+1/2)} = \frac{4^{2k}}{\sqrt{\pi}} \frac{(2k)!}{(4k)!},
\]

and whence the corollary is an immediate consequence of Theorem 3.6. \( \Box \)

We expect that several generalizations of the above results are valid and we state here some related conjectures.

**Conjecture 3.8.** For each positive integer \( m \), \( E_{2m,1/2}(x) \in \mathcal{L}-\mathcal{P}^+ \).

Extending the above considerations, in a private communication, M. Charalambides conjectured the following.

**Conjecture 3.9 (Charalambides).** If \( m \) and \( n \) are positive integers, then

\[
L_{m,n} = \left\{ \frac{(mk)! (nk)!}{((m+n+1)k)!} \right\}_{k=0}^{\infty}
\]

is a multiplier sequence and whence

\[
\varphi_{m,n}(x) := \sum_{k=0}^{\infty} \frac{\Gamma(mk+1) \Gamma(nk+1) x^k}{\Gamma((m+n+1)k+1) k!} \in \mathcal{L}-\mathcal{P}^+.
\]

Notice that \( \varphi_{1,2}(x) = \varphi_{2,1}(x) \in \mathcal{L}-\mathcal{P}^+ \) by Theorem 3.6. Conjecture 3.9 suggests the following problem involving the Fox–Wright functions.

**Problem 3.10.** Consider

\[
_2\Psi_1(x) = \sum_{k=0}^{\infty} \frac{\Gamma(ak+1) \Gamma(bk+1) x^k}{\Gamma(c k + d + 1) k!}.
\]

(3.7)
where \( a, b, c, d \geq 0 \) and \( c \geq a + b \). Under what additional restrictions on the parameters \( a, b, c, d \) is it true that \( 2 \Psi_1(x) \in L^{-\mathcal{P}+} \)?

The special case of (3.6) in Conjecture 3.9, when \( n = 0 \), is intriguing and appears to be tractable. (Equivalently, in (3.7) of Problem 3.10, set \( a = m, b = 0, c = m + 1 \) and \( d = -1 \), where \( m \) is a nonnegative integer.) Then we obtain a sequence of (entire) Fox–Wright functions which, in general, are not Mittag-Leffler functions:

\[
fm(x) := \sum_{k=0}^{\infty} \frac{\Gamma(mk + 1)}{\Gamma((m+1)k + 1)} \frac{x^k}{k!}, \quad m = 0, 1, 2, \ldots \quad (3.8)
\]

Observe that \( f_0(x) = J_0(2\sqrt{x}) \) is the Bessel function (which has only real zeros) and \( f_1(x) = \cosh(\sqrt{x}) \in L^{-\mathcal{P}+} \). For arbitrary positive integers \( m \geq 2 \), \( fm(x) \) is an entire function of order 1/2 (see the well-known formula [21, p. 4] or [1, p. 9]) and we can express \( fm(x) \) in terms of the generalized hypergeometric functions.

**Proposition 3.11.** For each positive integer \( m \geq 2 \), we have

\[
f_m(x) = \sum_{k=0}^{m-1} \binom{m}{k} \frac{1}{\Gamma(n-k+a)} \frac{1}{\Gamma(k+b)} \frac{x^k}{k!}, \quad m = 0, 1, 2, \ldots \quad (3.9)
\]

**Proof.** For \( m = 2 \) and \( m = 3 \), the Legendre duplication \((\sqrt{\pi} \Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2))\) and triplication \((2\pi \Gamma(3x) = 3^{3x-1/2}\Gamma(x)\Gamma(x+1/3)\Gamma(x+2/3))\) formulae [31, p. 23] yield, respectively,

\[
f_2(x) = \sum_{k=2}^{4} \binom{4}{2} \frac{1}{\Gamma(k+3)} \frac{1}{\Gamma(k+2)} \frac{1}{\Gamma(k+1)} \frac{2}{\Gamma(k+4)} \frac{4x^k}{k!} \quad \text{and} \quad f_3(x) = \sum_{k=3}^{8} \binom{8}{3} \frac{1}{\Gamma(k+3)} \frac{1}{\Gamma(k+4)} \frac{2}{\Gamma(k+5)} \frac{3}{\Gamma(k+6)} \frac{27x^k}{k!}. \quad (3.10)
\]

Finally, using the Gauss multiplication theorem [31, p. 26], together with an induction argument yields the desired hypergeometric representation of \( fm(x) \). \( \Box \)

We next proceed to show that the entire function \( f_2(x) \) has only real, negative zeros. Our proof will be based on the following lemma.

**Lemma 3.12.** If \( a \) and \( b \) are positive real numbers, then for any positive integer \( n \),

\[
s_n(a, b) := \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\Gamma(n-k+a)\Gamma(k+b)} = \frac{\Gamma(2n+a+b-1)}{\Gamma(n+a)\Gamma(n+b)\Gamma(n+a+b-1)}. \quad (3.11)
\]

**Proof.** We proceed to prove (3.11) by induction. By repeated application of the fundamental relation \( \Gamma(z+1) = z\Gamma(z) \), where \( z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \), one can readily verify (3.11) for all positive numbers \( a \) and \( b \), when \( n = 1 \) and \( n = 2 \). Next, suppose that (3.11) holds for a positive integer \( n \), and consider
\[ s_{n+1}(a, b) = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{\Gamma(n+1-k+a)\Gamma(k+b)} \]
\[ = \sum_{k=0}^{n} \left( \binom{n}{k} + \binom{n}{k-1} \right) \frac{1}{\Gamma(n+1-k+a)\Gamma(k+b)} + \frac{1}{\Gamma(a)\Gamma(n+1+b)} \]
\[ = s_n(a+1, b) + \left( s_n(a, b+1) - \frac{1}{\Gamma(a)\Gamma(n+1+b)} \right) + \frac{1}{\Gamma(a)\Gamma(n+1+b)} \]
\[ = \frac{\Gamma(2n+a+b)}{\Gamma(n+a+1)\Gamma(n+a+b)} + \frac{\Gamma(2n+a+b)}{\Gamma(n+a)\Gamma(n+b+1)\Gamma(n+a+b)} \]
\[ = \frac{1}{\Gamma(n+a+1)\Gamma(n+b+1)\Gamma(n+a+b)}. \]

Thus, the proof of the lemma is complete. \(\square\)

In the special case when \(a\) and \(b\) are positive integers and \(a = b\), formula (3.11) is known (see, for example, [15, formula (3.28)]).

**Theorem 3.13.** The entire function

\[ f_2(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{(3k)!} \frac{x^k}{k!} \in \mathcal{L}-\mathcal{P}^+. \] (3.12)

In particular, the sequence \(\{(2k)!/(3k)!\}_{k=0}^{\infty}\) is a meromorphic Laguerre multiplier sequence.

**Proof.** Since \(1/\Gamma(x) \in \mathcal{L}-\mathcal{P}(-\infty, 0]\) (cf. Theorem 2.4), it follows that the sequences

\[ T_1 := \left\{ \frac{1}{\Gamma(k+4/3)} \right\}_{k=0}^{\infty} \text{ and } T_2 := \left\{ \frac{1}{\Gamma(k+2/3)} \right\}_{k=0}^{\infty} \]

are CZDS (cf. Definition 2.3) and whence

\[ h_1(x) := T_1[e^x] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+4/3)} \frac{x^k}{k!} \in \mathcal{L}-\mathcal{P}^+ \text{ and } \]
\[ h_2(x) := T_2[e^x] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+2/3)} \frac{x^k}{k!} \in \mathcal{L}-\mathcal{P}^+. \] (3.13)

Now it follows that the entire function

\[ h_3(x) := h_1(x)h_2(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\Gamma(n-k+4/3)} \frac{1}{\Gamma(k+2/3)} \right) \frac{x^n}{n!} \in \mathcal{L}-\mathcal{P}^+. \] (3.15)
Setting $a = 4/3$ and $b = 2/3$ in Lemma 3.12, we see that

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{\Gamma(n-k+4/3)\Gamma(k+2/3)} = \frac{\Gamma(2n+1)}{\Gamma(n+4/3)\Gamma(n+2/3)\Gamma(n+1)},$$

and consequently (using (3.15)) $h_3(x)$ can be expressed as

$$h_3(x) = \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)}{\Gamma(n+4/3)\Gamma(n+2/3)\Gamma(n+1)} \frac{x^n}{n!} \in \mathcal{L}-\mathcal{P}^+. \quad (3.16)$$

Since $\varphi(x) := x + 1/3 \in \mathcal{L}-\mathcal{P}^+$, the sequence $T_3 = \{n + 1/3\}_{n=0}^{\infty}$ is a CZDS. Thus it follows that

$$h_4(x) := T_3[h_3(x)]$$

$$= \sum_{n=0}^{\infty} \frac{(n + 1/3)\Gamma(2n+1)}{\Gamma(n+4/3)\Gamma(n+2/3)\Gamma(n+1)} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(n + 1/3)\Gamma(2n+1)}{(n+1/3)\Gamma(n+1/3)\Gamma(n+2/3)\Gamma(n+1)} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)}{\Gamma(n+1/3)\Gamma(n+2/3)\Gamma(n+1)} \frac{x^n}{n!} \in \mathcal{L}-\mathcal{P}^+. \quad (3.17)$$

Using the Pochhammer notation [31, p. 23], we have that

$$\Gamma(n + 1/3) = \Gamma(1/3)(1/3)_n \quad \text{and} \quad \Gamma(n + 2/3) = \Gamma(2/3)(2/3)_n.$$ 

Also by the Legendre duplication formula [31, p. 24]

$$\frac{\Gamma(2n+1)}{\Gamma(n+1)} = \frac{2n\Gamma(2n)}{n\Gamma(n)} = 2 - \frac{1}{\sqrt{\pi}} 2^{2n-1} \frac{\Gamma(n)\Gamma(n+1/2)}{\Gamma(n)} = \frac{4^n}{\sqrt{\pi}} \Gamma(n + 1/2)$$

$$= 4^n \left( \frac{1}{2} \right)_n,$$

where we have used the familiar fact that $\Gamma(1/2) = \sqrt{\pi}$. Hence, with the aid of the foregoing calculations and Proposition 3.11 (see (3.10)), we can express $h_4(x)$ in the following form:

$$h_4(x) = \frac{1}{\Gamma(1/3)\Gamma(2/3)} \sum_{n=0}^{\infty} \frac{\Gamma(1/2)_n}{(1/3)_n(2/3)_n} \frac{(4x)^n}{n!} = \frac{1}{\Gamma(1/3)\Gamma(2/3)} f_2(27x)$$

$$= \frac{1}{\Gamma(1/3)\Gamma(2/3)} {}_1F_2 \left( \frac{1}{2}; \frac{1}{3}, \frac{2}{3}; 4x \right).$$

Since $h_4(x) \in \mathcal{L}-\mathcal{P}^+$, $f_2(x) \in \mathcal{L}-\mathcal{P}^+$. Finally, the second assertion of the theorem is clear, since $\psi(x) = \Gamma(2x+1)/\Gamma(3x+1)$ is a meromorphic function and the sequence $\{(2k)!/(3k)!\}_{k=0}^{\infty}$ is a multiplier sequence. \qed
We remark that the functions \( h_1(x) \) and \( h_2(x) \) can be readily expressed in terms Bessel functions and it is known that these Bessel functions have only real zeros. We preferred to use a straightforward application of multiplier sequences to give a direct and simple proof of (3.13) and (3.14).

The method used to prove Theorem 3.13 yields the following general result.

**Theorem 3.14.** If \( a \) and \( b \) be positive real numbers, then the entire function

\[
\varphi_{a,b}(x) := \sum_{k=0}^{\infty} \frac{\Gamma(2k + a + b - 1)}{\Gamma(k + a)\Gamma(k + b)\Gamma(k + a + b - 1)} \frac{x^k}{k!} \in \mathcal{L}-\mathcal{P}^+.
\]

In particular, the sequence

\[
\left\{ \frac{\Gamma(2k + a + b - 1)}{\Gamma(k + a)\Gamma(k + b)\Gamma(k + a + b - 1)} \right\}_{k=0}^{\infty}
\]

is a meromorphic Laguerre multiplier sequence.

It is an immediate consequence of Laguerre’s theorem (see part (3) of Theorem 2.4) that if \( \varphi \in \mathcal{L}-\mathcal{P}(-\infty, 0] \), then for any positive real number \( \alpha \), the sequence \( \{\varphi(\alpha k)\}_{k=0}^{\infty} \) is again a multiplier sequence. An analogous result does not hold, in general, for meromorphic Laguerre multiplier sequences, as the following example shows. By Theorem 3.13, \( \psi(x) = \Gamma(2x + 1)/\Gamma(3x + 1) \) is a meromorphic function and the sequence \( T := \{\psi(k)\}_{k=0}^{\infty} \) is a multiplier sequence. But the sequence \( T_\alpha := \{\psi(\alpha k)\}_{k=0}^{\infty} \) is not a multiplier sequence if, for example, \( \alpha = 1/10 \). Indeed, numerical work shows that when \( \alpha = 1/10 \), the polynomial \( T_\alpha[(1 + x)^5] \) has two nonreal zeros. While we do not know the set of all values of \( \alpha \) for which \( T_\alpha \) is a multiplier sequence, with the aid of following theorem of G. Pólya, we are able to show that \( T_\alpha \) is a multiplier sequence whenever \( \alpha \) is a nonnegative integer.

**Theorem 3.15** (Pólya [27, p. 320]). Let \( q, r \) and \( n \) be positive integers such that \( qr \leq n < (q + 1)r \). If the polynomial

\[
p(x) := \sum_{k=0}^{n} b_k \frac{x^k}{k!} = b_0 + b_1 \frac{x}{1!} + b_2 \frac{x^2}{2!} + \cdots + b_n \frac{x^n}{n!}, \quad b_k \in \mathbb{R},
\]

has only real negative zeros, then the polynomial

\[
\hat{p}(x) := b_0 + b_r \frac{x^r}{r!} + b_2 \frac{x^2}{2!} + \cdots + b_q \frac{x^q}{q!}
\]

also has only real negative zeros.

**Corollary 3.16.** For each positive integer \( m \), the entire function

\[
\varphi_m(x) := \sum_{k=0}^{\infty} \frac{(2mk)! x^k}{(3mk)! k!}
\]
has only real negative zeros (i.e., $\varphi_m(x) \in \mathcal{L}-\mathcal{P}^+$) and whence the sequence $\{(2mk)!/(3mk)!\}_{k=0}^{\infty}$ is a meromorphic Laguerre multiplier sequence.

**Proof.** Fix a positive integer $m$. By Theorem 3.13, the function

$$f_2(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{(3k)!} \frac{x^k}{k!} \in \mathcal{L}-\mathcal{P}^+$$

and hence the associated Jensen polynomials (cf. (2.5))

$$g_{nm}(x) := \sum_{k=0}^{nm} \frac{(nm)!}{(nm-k)!} \frac{(2k)!}{(3k)!} \frac{x^k}{k!}, \quad n = 1, 2, 3, \ldots,$$

have only real negative zeros. Thus, by Theorem 3.15, the polynomials

$$\hat{g}_{nm}(x) := \sum_{k=0}^{nm} \frac{(nm)!}{(nm-mk)!} \frac{(2mk)!}{(3mk)!} \frac{x^k}{k!}, \quad n = 1, 2, 3, \ldots,$$

also have only real negative zeros. Let $h_{nm}(x) := \hat{g}_{nm}(x^m)$. The zeros of $h_{nm}(x)$ lie on $m$ rays, $R_1, \ldots, R_m$, where the ray $R_k$ emanates from the origin and passes through the point $\exp((\pi i + 2\pi ik)/m)$, $k = 1, 2, \ldots, m$. Then $h_{nm}(x/nm) \to \varphi_m(x^m)$, as $n \to \infty$, uniformly on compact subsets of $\mathbb{C}$ and the zeros of $\varphi_m(x^m)$ also lie on the $m$ rays, $R_1, \ldots, R_m$. Therefore, using the transformation $x \to \sqrt[m]{x}$, we conclude that $\varphi_m(x) \in \mathcal{L}-\mathcal{P}^+$ (cf. (2.6) and (2.7)).

Motivated, in part, by the results of Proposition 3.11, Theorem 3.13 and Corollary 3.16, we conclude this paper with following conjecture.

**Conjecture 3.17.** For each positive integer $m$, the Fox–Wright function $f_m(x)$ (cf. (3.8)) has only real negative zeros and whence the sequence $\left\{\Gamma(mk+1)/\Gamma((m+1)k+1)\right\}_{k=0}^{\infty}$ is a meromorphic Laguerre multiplier sequence.

**References**