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PLACES ON *-FIELDS AND
THE REAL HOLOMORPHY RING

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1. Introduction. By a *-field, we mean a skew field $D$ together with an
involution *. Many ideas and results from the theory of formally real fields
can be extended to *-fields [C1-4; H1-2]. In this paper, our primary interest
will be in real places and the real holomorphy ring (intersection of all real
valuation rings). This should be thought of as a prelude to extending the
concepts of real algebraic geometry to *-fields. For an ordered skew field $D$,
A. A. Albert showed that the center of $D$ must be algebraically closed in $D$.
Essentially the same is true of *-fields. Chacron (cf. [H2]) has shown that
the existence of a *-ordering (definition below) on $(D, *)$ implies that either
$(D, *)$ is a standard quaternion algebra or the center of $D$ is algebraically
closed in $D$ (i.e., every noncentral element is transcendental over the center).

Briefly, the paper is organized as follows. After a quick overview of the
entire paper, this section introduces the basic definitions and notation needed
for the remainder. Section two studies places in the context of *-fields. The
emphasis will be on places into the real quaternions $\mathbb{H}$, which takes the role of
$\mathbb{R}$ in the theory of formally real fields. Since we would like to have a unique
$\mathbb{H}$-valued place corresponding to an ordering, we introduce an equivalence
relation on the set of $\mathbb{H}$-valued places. One of the main results of §2 is

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that the equivalence classes depend only on the action of the place on the symmetric elements of $D$, namely $S(D) = \{d \in D \mid d^* = d\}$.

The third section works with the set of all real places, giving it a topology and studying the evaluation mappings induced by symmetric elements of the holomorphy ring. It is also shown how the places induce a decomposition of the space of orderings. Section 4 deals with a specific place, namely the finest one through which all other places factor. This provides techniques for dealing with local behavior.

The final section analyzes some of the properties of the real holomorphy ring. The main results show that it is a noncommutative Prüfer domain and that its elements are precisely those for which there is some positive integer which bounds the element with respect to every ordering. Homomorphisms on the holomorphy ring are shown to correspond to real places.

In addition to $S(D)$, we require notation for several other distinguished subsets of $(D, \ast)$. The center of $D$ will be denoted by $Z_D$. For any subset $S \subseteq D$, the notation $S^\times$ will be used for $S \setminus \{0\}$, the subset of nonzero elements of $S$, except where explicitly stated that it denotes the set of units of a ring. By $[D^\times, S(D)^\times]$, we shall mean the multiplicative subgroup of $D^\times$ generated by all mixed commutators $[d, s] = dsd^{-1}s^{-1}$ for $d \in D^\times$, $s \in S(D)^\times$. Elements of the form $dd^*$, $d \in D^\times$, will be called norms.

**DEFINITION 1.1:** We write $\Sigma$ or $\Sigma(D)$ for the set of sums of products of nonzero norms and elements of $[D^\times, S(D)^\times]$; and we write $S(\Sigma) = S(\Sigma(D))$ for $\Sigma(D) \cap S(D)$.

These two sets, $\Sigma$ and $S(\Sigma)$, play the role of sums of squares in the theory of formally real fields and thus are extremely important to our theory. It is not hard to see that, when $0 \notin \Sigma$, then $\Sigma$ is a normal subgroup of $D^\times$. 


An extended $\ast$-ordering (called "strong ordering" in [H2]) is a $\ast$-closed subset $P$ of $D$ satisfying $P + P \subseteq P$, $1 \in P$, $dP \ast \subseteq P$, for all $d \in \mathcal{D}^\times$; $P \cup -P \supseteq S(D)^\times$; $P \cap -P = \emptyset$ and $P \cdot P \subseteq P$. A $\ast$-ordering is the set of symmetric elements in an extended $\ast$-ordering. Each $\ast$-ordering $P$ is contained in a maximal extended $\ast$-ordering, denoted $P^\ast$ [C1]. The intersection of all extended $\ast$-orderings is $\Sigma(D)$, and so the intersection of all $\ast$-orderings is $S(\Sigma)$. Furthermore, $\ast$-orderings exist iff $-1 \notin S(\Sigma)$ [H2]. We denote the set of all $\ast$-orderings of $D$ by $\mathcal{X}_D$.

A subring $A$ of $D$ which contains either $d$ or $d^{-1}$ for each $d \in \mathcal{D}^\times$, will be called a total subring of $D$. If it is also invariant (i.e. $dAd^{-1} = A$ for all $d \in \mathcal{D}^\times$), then we call $A$ a valuation ring. A $\ast$-valuation ring is a valuation ring closed under the involution $\ast$, or equivalently, a total subring containing $d^*d^{-1}$ for each $d \in \mathcal{D}^\times$ [H1]. The associated valuation $v$ is called a $\ast$-valuation and is characterized by the fact that $v(d) = v(d^*)$ for any $d \in \mathcal{D}^\times$. All valuations will be written additively. Given a $\ast$-valuation $v$, the valuation ring, maximal ideal, group of units, value group and residue $\ast$-field will be denoted by $A_v$, $m_v$, $U_v$, $\Gamma_v$ and $\mathcal{D}_v$, respectively. Following [C2; C4], a $\ast$-valuation $v$ will be called real if $[\mathcal{D}_v, S(\mathcal{D})^\times]$ (contained in $U_v$) has image contained in some extended $\ast$-ordering of the residue $\ast$-field $\mathcal{D}_v$. This condition guarantees that all $\ast$-orderings of $\mathcal{D}_v$ lift to $D$ [C4, Theorem 2.2]. We shall also say that the associated valuation ring $A_v$ is real.

A $\ast$-valuation $v$ is said to be compatible with a $\ast$-ordering $P$ if $0 < a \leq b$ with respect to $P$ implies that $v(a) \geq v(b)$ in the value group. We shall also say $v$ is compatible with $A_v$. Associated with any $\ast$-valuation we have a $\ast$-place, namely a place $\pi$ which preserves the involution and such that $\pi(d^*d^{-1}) \neq \infty$, for all $d \in \mathcal{D}^\times$ [H1, p.22]. We shall say that a $\ast$-place $\pi$ is compatible with a $\ast$-ordering $P$ if its associated valuation ring is. When the valuation is not explicit, we shall write $A_\pi$, $U_\pi$, $\mathcal{D}_\pi$, etc. for the objects
associated with the place. A \*-place \( \pi \) will be called real iff the associated valuation ring is real. This is equivalent to checking that \( \pi([D^\times, S(D)^\times]) \) is contained in some (equivalently all \([C_4, \text{Proposition 2.1}]\)) extended \*-ordering of the residue \*-field.

Finally, we can define one of the main objects of study for this paper. The real holomorphy ring of \((D, \ast)\) is the subring \( \mathcal{H}(D) \) equal to the intersection of all real \*-valuation rings of \( D \). This generalizes the case where \( \ast = \text{identity} \), which has a rapidly growing literature (cf. \([L, \S 9]\)).

2. \( \mathbb{H} \)-valued places. For any \*-ordering \( P \), we have the order valuation \( \upsilon_p \), a \*-valuation compatible with \( P \) and having associated \*-valuation ring

\[ A(P) = \{ d \in D \mid 0 \leq dd^* < n \text{ with respect to } P \text{ for some } n \in \mathbb{Z} \}, \]

maximal ideal

\[ \mathfrak{m}_p = \{ d \in D \mid 0 \leq dd^* < q, \text{ for all } q \in \mathbb{Q}^+ \} \]

and residue \*-field \( \bar{D} = A(P)/\mathfrak{m}_p \). (The induced involution on \( D \) will also be denoted by \( \ast \).) The induced \*-ordering \( \bar{P} \) on \( \bar{D} \) makes \( \bar{D} \) into an archimedean ordered \*-field, \( \ast \) and order isomorphic to a subfield of \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} [H_1; H_2] \). Since \( \mathbb{R} \) and \( \mathbb{C} \) can be thought of as subfields of \( \mathbb{H} \), we obtain an induced \*-place \( \pi_p : (D, \ast) \rightarrow \mathbb{H} \cup \{ \infty \} \). The valuation ring is a real \*-valuation ring.

Unlike the situation for \( \mathbb{R} \)-valued places on formally real fields, the \*-place \( \pi_p \) is not uniquely determined by \( P \). Indeed, we obtain other choices by composing \( \pi_p \) with any automorphism of \( \mathbb{H} \) which fixes \( \mathbb{R} \). For example, \( \mathbb{C} \) has a unique \*-ordering \( \mathbb{R}^+ \), but there exist two \*-places onto \( \mathbb{C} \) (identity and conjugation) and many into \( \mathbb{H} \).
Let $M_D$ be the set of equivalence classes of $H$-valued real $*$-places on a $*$-field $D$ modulo the equivalence relation $\pi_1 \sim \pi_2$ iff there exists a $*$-automorphism $\phi$ of $H$ such that

$$
\begin{array}{ccc}
D & \xrightarrow{\pi_1} & H \cup \{\infty\} \\
\parallel & & \downarrow \phi \\
D & \xrightarrow{\pi_2} & H \cup \{\infty\}
\end{array}
$$

commutes. Note that this differs from the usual notion of equivalence of places in that the residue $*$-fields may be isomorphic via an isomorphism which does not extend to $H$.

We shall write $c(H)$ for the equivalence class in $M_D$ containing an $H$-valued real $*$-place $\pi$. We shall eventually see that the importance of these equivalence classes is that each is the set of all $W$-valued $*$-places under which some $*$-ordering $P$ of $D$ is nonnegative.

**Proposition 2.1.** Let $\pi: D \to H \cup \{\infty\}$ be a $*$-place and let $P \in X_D$. If $\pi(P) \geq 0$ (where by convention, $\infty > 0$), then $\pi$ and $P$ are compatible.

**Proof:** By [C3, Theorem 4.2], $\pi$ and $P$ are compatible iff the pushdown $\hat{P}$ on the residue $*$-field is a $*$-ordering. Since $P = \pi(P \cap U_\pi) \subseteq \mathbb{R}^+$, this holds. $\blacksquare$

Note that we do not use $\pi(P) \geq 0$ as the definition of compatibility. We find it better to follow Lam [L] in making the definition for places in agreement with the definition for valuations. This means that the residue $*$-field may have embeddings into $H$ in which $\hat{P}$ is not contained in $\mathbb{R}^+$ and these induce places also considered compatible with $P$. We can strengthen the previous proposition considerably.
Proposition 2.2. Let $P \in X_D$. If a $*$-place $\pi : D \to \mathbb{H} \cup \{\infty\}$ satisfies $\pi(P) \geq 0$, then its valuation ring is $A(P)$.

Proof: Assume $\pi(P) \geq 0$ and write $A$, $m$, $U$ and $v$ for the valuation ring, maximal ideal, group of units and $*$-valuation associated with $\pi$. Let $a \in \pi^{-1}(\mathbb{R}^+) \cap S(D)$, so in particular, $a \in U$. If $a \notin P$, then $-a \in P$ and $0 \leq \pi(-a) = -\pi(a)$, contradicting our choice of $a$. Therefore

$$P \cap U = \pi^{-1}(\mathbb{R}^+) \cap S(D)$$

Using (2.3) plus the fact that $\pi(1 + m) = 1$, we obtain $1 + (m \cap S(D)) \subseteq P$, so that $v$ is compatible with $P$, and, in particular, $A$ is convex with respect to $P$ [C3, Theorem 4.2]. Since $A$ contains $\mathbb{Q}$, this implies $A$ contains $dd^*$ for any $d \in A(P)$. But, as $A$ is a valuation ring, either $d \in A$ or $d^{-1} \in A$; if $d^{-1} \in A$, then $d^* = d^{-1}(dd^*) \in A$, and hence $d \in A$ since $A$ is closed under $*$. Thus $A$ contains $A(P)$. On the other hand, assume $d \in A$ and let $n \in \mathbb{Z}^+$ such that $\pi(dd^*) < n$. Then the unit $n - dd^*$ lies in $P$ by (2.3), so $d \in A(P)$. Therefore $A = A(P)$. \[\square\]

Remark 2.4. We note that one fact established in the preceding proof was that if a $*$-valuation ring $A$ is compatible with a $*$-ordering $P$, then $A(P) \subseteq A$. Thus the rings $A(P)$ are the minimal real $*$-valuation rings. Therefore we can write

$$\mathcal{H}(D) = \bigcap_{P \in X_D} A(P)$$

The next result will be very important in dealing with elements of $\mathcal{M}_D$.

Theorem 2.5. In $\mathcal{M}_D$, $cl(\pi_1) = cl(\pi_2)$ iff $\pi_1$ and $\pi_2$ agree on $S(D)$.

Proof: If $cl(\pi_1) = cl(\pi_2)$, then $\pi_1$ and $\pi_2$ must be equal on any element of $D$ mapping into $\mathbb{R} \cup \{\infty\}$. In particular, this applies to $S(D)$. Now assume $\pi_1$ and $\pi_2$ agree on $S(D)$. 
Let $A = \pi_1^{-1}(H) = \{d \in D \mid 0 \leq \pi_1(dd^*) < \infty\} = \{d \in D \mid 0 \leq \pi_2(dd^*) < \infty\}$ be the common valuation ring of $\pi_1$ and $\pi_2$. Let $\mathfrak{m}$ be its maximal ideal and $U$ its group of units. If $d \not\in A$, then $\pi_1(d) = \pi_2(d) = \infty$ so we need not be concerned with such elements. Let $d \in A$ and write $d = s + k$ where $s = (d + d^*)/2$ is the symmetric part and $k = (d - d^*)/2$ is the skew part. Since $A$ is $*$-closed, $s$ and $k$ both lie in $A$. We now distinguish three cases depending on the image $\mathcal{D} = \pi_1(A/\mathfrak{m})$ in $H$.

Case 1: $\mathcal{D} \subseteq \mathbb{R}$. Since $\pi_1$ preserves $*$, each skew element of $\mathcal{D}$ maps to 0 or $\infty$. Since $\pi_2$ can differ from $\pi_1$ only on nonsymmetric elements of $U$, we have $\pi_1 = \pi_2$.

Case 2: $\mathcal{D} \subseteq \mathbb{C}$ (or isomorphic commutative subfield of $H$), but $\mathcal{D} \not\subseteq \mathbb{R}$. We claim that $\pi_1$ either equals $\pi_2$ or equals $\pi_2$ followed by conjugation in $\mathbb{C}$, so that $\text{cl}(\pi_1) = \text{cl}(\pi_2)$. From the initial part of the proof, we know that it suffices to consider skew units; assume $x \in U$ with $x^* = -x$ and $\pi_1(x) \neq \pi_2(x)$. Since $x^2 \in S(\mathcal{D})$, we have $\pi_1(x)^2 = \pi_2(x)^2$ in $\mathbb{R}$, so that $\pi_1(x) = -\pi_2(x)$. If $\pi_1$ and $\pi_2$ are not related by conjugation, there exists $y \in U$, $y^* = -y$, with $\pi_1(y) = \pi_2(y)$. But then $\pi_1(xy + yx) = -\pi_2(xy + yx)$. As $xy + yx \in S(\mathcal{D})$, we must have $xy + yx \in \mathfrak{m}$. But then $xyx^{-1}y^{-1} \equiv -1$ modulo $\mathfrak{m}$ and the valuation is not real, a contradiction.

Case 3: $\mathcal{D}$ is not commutative. By a result of Dieudonné [D1, Lemma 1; D2, §14], either $\mathcal{D}$ is generated by its symmetric elements (not possible since they are all central) or $\mathcal{D}$ is a generalized quaternion algebra $(\frac{a,b}{F})$ with standard involution, where $F = \mathcal{D} \cap \mathbb{R}$ and $a, b \in F$. Let $\bar{x}, \bar{y}$ be generators of $\mathcal{D}$ with $\bar{x}^2 = a, \bar{y}^2 = b$ and let $x, y \in D$ be skew elements with $\pi_1(x) = \bar{x}$, $\pi_1(y) = \bar{y}$. Now $\pi_2(A/\mathfrak{m}) = (\frac{a,b}{F})$ also, with generators $\pi_2(x)$ and $\pi_2(y)$. Using the fact that $\pi_1$ and $\pi_2$ agree on $S(\mathcal{D})$, the vector space isomorphism defined by $1 \mapsto 1$, $\pi_1(x) \mapsto \pi_2(x)$, $\pi_1(y) \mapsto \pi_2(y)$ and $\pi_1(xy) \mapsto \pi_2(xy)$ is
easily seen to be an automorphism of $\bar{D}$. By the Skolem-Noether theorem, it is an inner automorphism, so we have $\pi_1(d) = \alpha \pi_2(d) \alpha^{-1}$ for $d = x, y$ or $xy$ and some $\alpha \in \bar{D} \subseteq H$. We claim that conjugation by $\alpha$ is the desired automorphism of $H$ connecting $\pi_1$ and $\pi_2$. If not, there exists $u \in U$, $u^* = -u$, such that $\pi_1(u) \neq \alpha \pi_2(u) \alpha^{-1}$; say $\pi_1(u) = \alpha \pi_2(u) \alpha^{-1} + \varepsilon$ for some $\varepsilon \neq 0$ in $\bar{D}$. Since a nonzero $\varepsilon$ cannot anticommute with all of $x, y$ and $xy$, we may assume $\varepsilon x + \bar{x} \varepsilon \neq 0$. Then we have

\[
\pi_1(xu + ux) = (\alpha \pi_2(x) \alpha^{-1})(\alpha \pi_2(u) \alpha^{-1}) + \varepsilon x + (\alpha \pi_2(x) \alpha^{-1})(\alpha \pi_2(u) \alpha^{-1}) + \bar{x} \varepsilon
\]

\[
= \alpha \pi_2(xu + ux) \alpha^{-1} + \bar{x} \varepsilon + \varepsilon x
\]

\[
= \pi_2(xu + ux) + (\bar{x} \varepsilon + \varepsilon x)
\]

\[
\neq \pi_2(xu + ux),
\]

a contradiction of $xu + ux \in S(D)$. Therefore $\mathcal{c}(\pi_1) = \mathcal{c}(\pi_2)$. ■

**Corollary 2.6.** Let $P \in X_D$ and let $\pi_1, \pi_2$ be $H$-valued $*$-places.

1. If $\pi_1(P) \geq 0$ and $\pi_2(P) \geq 0$, then $\mathcal{c}(\pi_1) = \mathcal{c}(\pi_2)$ in $\mathcal{M}_D$.

2. If $\mathcal{c}(\pi_1) = \mathcal{c}(\pi_2)$ and $\pi_1(P) \geq 0$, then $\pi_2(P) \geq 0$.

**Proof:** (1) Assume $\mathcal{c}(\pi_1) \neq \mathcal{c}(\pi_2)$. By Theorem 2.5, there exists $a \in S(D)$ such that $\pi_1(a) \neq \pi_2(a)$. By Proposition 2.2, we may assume $a$ lies in their common valuation ring $A(P)$, and thus we may assume $\pi_1(a) < \pi_2(a)$ in $\mathbb{R}$. Choose $q \in \mathbb{Q}$ such that $\pi_1(a) < q < \pi_2(a)$. Consider now the symmetric element $a - q \in P \cup -P$. The fact that $\pi_1(a - q) < 0$ and $\pi_2(a - q) > 0$ contradicts the hypothesis that $\pi_i(P) \geq 0$ for both $i = 1, 2$.

(2) Since $P \subseteq S(D)$, the claim follows from Theorem 2.5. ■

From this proposition, we see that, given a $*$-ordering $P$ in $X_D$, there exists a unique equivalence class in $\mathcal{M}_D$ consisting exactly of all $H$-valued
*-places $\pi$ with $\pi(P) \geq 0$. We denote this element of $\mathcal{M}_D$ by $\lambda(P)$ and write $\lambda$ for the induced function $\lambda : X_D \to \mathcal{M}_D$.

**Proposition 2.7.** The mapping $\lambda : X_D \to \mathcal{M}_D$ is surjective.

**Proof:** Let $\pi : D \to H \cup \{\infty\}$ be a *-place. The residue *-field $\bar{D}$ has a *-ordering $\bar{P} = \bar{D} \cap \mathbb{R}^+$. By [C4, Theorem 2.2], there exists a *-ordering $P \in X_D$ such that $\pi(P \cap U_\pi) = \bar{P}$. In particular, $\pi(P) \geq 0$, so $c\ell(\pi) = \lambda(P)$.

Any $H$-valued real *-place comes from a *-ordering by the previous proposition and hence carries the mixed commutator group $[D^\times, S(D)^\times]$ to 1 by [H2, Theorem 5.6].

The inverse image under $\lambda$ of an element of $\mathcal{M}_D$ is the set of all *-orderings in $X_D$ which push down to the *-ordering $P$ of the residue *-field defined in the previous proof. This set is completely described by [C4, Theorem 2.2] or [C2, Theorem 3.4].

**3. Topological spaces of $H$-valued places.** The set $X_D$ has a topology on it making it a Boolean space (compact, Hausdorff and totally disconnected) with a subbasis for the clopen sets consisting of all sets of the form

$$H(a) = \{ P \in X_D \mid a \in P \}, \quad a \in S(D)^\times$$

(see [C2, §4] for details). We give $\mathcal{M}_D$ the quotient topology induced by this topology on $X_D$. Since $X_D$ is compact, $\mathcal{M}_D$ is a compact space also. Our goal for this section is to see how much more can be said about this topology on $\mathcal{M}_D$. The elements of $\mathcal{H}(D)$ will play an important role here. It will be useful to prove our theorems in somewhat greater generality as this can be done with essentially no extra work. We begin by recalling the concepts of preordering and extended preordering from [C3, §3].
An extended preordering is a \(*\)-closed subset \(T\) in \(D^*\) satisfying \(T + T \subseteq T\), \(dT^* \subseteq T\) for all \(d \in D^*\), \(\Sigma(D) \subseteq T\), \(-1 \notin T\) and \(T \cdot T \subseteq T\). A preordering is the set of symmetric elements in an extended preordering. An algebraic description can be found in \([C3]\). It is also characterized by the fact that it equals the intersection of all \(*\)-orderings containing it. For any preordering \(T\), the minimal extended preordering containing it will be denoted by \(T^e\) \([C3, \text{Theorem 3.6}]\). Associated with a preordering \(T\) we have

\[ X_T = \{ P \in X_D \mid T \subseteq P \}, \]

a Boolean subspace of \(X_D\) in the relative topology. We say \(T\) is compatible with a valuation \(v\) (or with the associated valuation ring \(A_v\)) if some \(P \in X_T\) is compatible with \(v\) and \(T\) is fully compatible with \(v\) (or \(A_v\)) if every \(P \in X_T\) is compatible with \(v\). We shall write \(\mathcal{M}_T\) for the image \(\lambda(X_T)\), a closed subspace of \(\mathcal{M}_D\). We say \(\lambda(P) \in \mathcal{M}_D\) is compatible with \(T\) if the associated valuation ring \(A(P)\) is compatible with \(T\).

We require the notion of a “wedge product” from \([C3, \S 4]\). Let \(v\) be a \(*\)-valuation on \((D, *)\), \(T\) a preordering of \(D\) and \(Q\) a preordering of the residue \(*\)-field \(D_v\) containing \(T\). Then we define

\[ T \wedge Q = \{ \sum t_i u_i \mid t_i \in T^e, u_i \in \Pi S(D), \text{ with } \bar{u}_i \in \bar{Q}^e \} \cap S(D) \]

The set \(T \wedge Q\) is a preordering of \(D\), fully compatible with \(v\), with \((T \wedge Q) \cap U_v\) mapping to \(Q\) in the residue \(*\)-field \([C3, \text{Theorem 4.4}]\). In particular, if \(\pi(T) \geq 0\), for \(Q\) we can use \(P_\pi\), the unique \(*\)-ordering \(D_\pi \cap R^+\) induced on the residue \(*\)-field \(D_\pi\) by its embedding in \(H\).

**Lemma 3.1.** Let \(\pi(T) \geq 0\) and set \(T_\pi = T \wedge P_\pi\). If \(P \in X_{T_\pi}\), then \(cl(\pi) = \lambda(P)\).

**Proof:** Since \(P \supseteq T_\pi\), we have \(P \supseteq \overline{T \wedge P_\pi} = P_\pi\), so \(\bar{P} = P_\pi\) and hence \(\pi(P) \geq 0\). Then \(cl(\pi) = \lambda(P)\) by definition. \(\blacksquare\)
PROPOSITION 3.2. For any preordering \( T \) and any real \( * \)-place \( \pi \), we have

\[
\ell(\pi) \in \mathcal{M}_T \iff \pi(T) \geq 0
\]

PROOF: By Proposition 2.7, there exists a \( * \)-ordering \( P \) such that \( \ell(\pi) = \lambda(P) \). If \( \ell(\pi) \in \mathcal{M}_T \), then \( T \subseteq P \). Since \( \pi(P) \geq 0 \), we have \( \pi(T) \geq 0 \). Conversely, assume \( \pi(T) \geq 0 \). Let \( P \) be any \( * \)-ordering containing \( T \land P_\pi \). By Lemma 3.1, \( \ell(\pi) = \lambda(P) \). Since \( P \supseteq T \land P_\pi \supseteq T \), we have \( \ell(\pi) \in \mathcal{M}_T \). \( \blacksquare \)

This proposition shows that the \( * \)-ordering \( P_\pi \) of the residue \( * \)-field depends only on the equivalence class of \( \pi \) in \( \mathcal{M}_T \).

The real holomorphy ring \( \mathcal{H}(D) = \bigcap_{P \in \mathcal{X}_D} A(P) \) is a special case of a class of rings associated with preorderings. For any preordering \( T \), we write \( A(T) \) for the intersection of all \( * \)-valuation rings compatible with \( T \). By Remark 2.4, we have

\[
A(T) = \bigcap_{P \in \mathcal{X}_T} A(P)
\]

If we take \( T \) to be the preordering \( S(\Sigma) \), then we recover the real holomorphy ring \( \mathcal{H}(D) \). Note that \( A(T) \) is an invariant subring of \( D \) since each \( A(P) \) is invariant.

LEMMA 3.3. Let \( T \) be a preordering. Let \( d \in D \), \( u \in [D^*, S(D)^*] \) and \( t \in T^c \cup \{0\} \). Then

1. \( (u + t)^{-1} \in A(T) \);
2. \( d(1 + dd^* + t)^{-1}, d^*(1 + dd^* + t)^{-1} \in A(T) \);
3. \( A(T) \) is a (right and left) Ore domain with \( D \) as its field of fractions.

PROOF: Let \( A \) be a \( * \)-valuation subring of \( D \) compatible with \( T \) having maximal ideal \( m \). For (1) and (2), it suffices to show that the elements lie in \( A \).
(1) If \((u + t)^{-1} \notin A\), then \(u + t \in m\). Pushing down to the residue \(*\)-field yields \(-\bar{u} \in \bar{T}\), a contradiction of compatibility with \(T\).

(2) If \(d \in A\), then \(d(1 + dd^* + t)^{-1}, d^*(1 + dd^* + t)^{-1}\) lie in \(A\) by (1). Now assume \(d \notin A\). Then \(d^{-1}, d^{-1}\) lie in \(A\) by (1). For the first element, a computation shows

\[
d(1 + dd^* + t)^{-1} = (d^{-1}, dd^*) + d^{-1}d^{-1} + d^{-1}td^{-1})^{-1}d^{-1},
\]

where \([d^{-1}, dd^*] \in [D^*, S(D)^*]\) and \(d^{-1}d^{-1} + d^{-1}td^{-1} \in T\), so again we are done by (1). Similarly, \(d^*(1 + dd^* + t)^{-1} = (1 + d^{-1}d^{-1} + d^{-1}td^{-1})^{-1}d^{-1}\) lies in \(A\) by (1).

(3) To show that \(A(T)\) is a right Ore domain, we must show that for nonzero \(a, b \in A(T)\), we have \(aA(T) \cap bA(T) \neq 0\) [Co2, p. 8]. Since \(A\) is invariant, \(b^{-1}ab \in A\) and so \(0 \neq ab = b(b^{-1}ab) \in aA(T) \cap bA(T)\). Similarly, \(A(T)\) is a left Ore domain. Its field of fractions is thus unique (up to isomorphism) and is equal to \(D\) because any \(d \in D\) can be written \([d(1 + dd^*)^{-1}]/(1 + dd^*)^{-1}\) as a quotient of two elements of \(A(T)\).

For each element \(a \in S(D)\), we have an evaluation map

\[
\hat{a} : M_T \to \mathbb{R} \cup \{\infty\} \subseteq H \cup \{\infty\},
\]

defined by \(\hat{a}(\mathcal{C}(\pi)) = \pi(a)\) for all \(\mathcal{C}(\pi) \in M_T\). Note that this is well-defined since all places in \(\mathcal{C}(\pi)\) map the symmetric elements of \(D\) into the symmetric elements of \(H\), viz. \(\mathbb{R}\), which is fixed by all \(*\)-automorphisms of \(H\).

**Theorem 3.4.** For \(a \in S(D)\), the evaluation map \(\hat{a} : M_T \to \mathbb{R} \cup \{\infty\}\) is continuous.

**Proof:** First consider the case where the image of \(\hat{a}\) is contained in \(\mathbb{R}\) (i.e. \(a \in A(T)\)). Because \(M_T\) has the quotient topology induced by \(\lambda : X_T \to\)
\( M_T \), it will suffice to show that the composite function \( f = \dot{\alpha} \circ \lambda : X_T \to \mathbb{R} \) is continuous. Consider the inverse image under \( f \) of the open interval \((q, \infty)\) for \( q \in \mathbb{Q} \). For any \( P \in X_T \), one easily obtains from the definitions that

\[ P \in f^{-1}(q, \infty) \iff a - q > q_0 \]

with respect to \( P \) for some positive \( q_0 \in \mathbb{Q} \), and therefore \( f^{-1}(q, \infty) = \bigcup_{q_0 \in \mathbb{Q}} H(a - q - q_0) \) is an open set in \( X_T \). For any real number \( r \), the set \( f^{-1}(r, \infty) = \bigcup_{q > r} f^{-1}(q, \infty) \) is also open. A similar argument shows \( f^{-1}(-\infty, r) \) is open for any \( r \in \mathbb{R} \). Since these intervals form a subbasis for the topology of \( \mathbb{R} \), we have shown that \( f \) is continuous.

Next consider an arbitrary element \( a \in S(D) \) and let \( \text{cl}(\pi) \in M_T \). If \( \pi(a) \neq \infty \), write \( a = bc^{-1} \), where \( b = a(1 + a^2)^{-1} \) and \( c = (1 + a^2)^{-1} \), both in \( \mathcal{A}(T) \) by Lemma 3.3. Then \( \hat{b} \) and \( \hat{c} \) are continuous on \( M_T \) by the preceding paragraph. Since \( \pi(a) \neq \infty \), we have \( \pi(c) \neq 0 \) and therefore \( \hat{a} = \hat{b}/\hat{c} \) is continuous at the point \( \text{cl}(\pi) \) in \( M_T \). On the other hand, if \( \pi(a) = \infty \), then \( a^{-1}(\pi) = 0 \), so the preceding case shows \( a^{-1} \) is continuous at \( \text{cl}(\pi) \).

Specifically, given any \( \varepsilon > 0 \), there exists a neighborhood \( N \) of \( \text{cl}(\pi) \) in \( M_T \) such that \( |\pi_0(a^{-1})| < \varepsilon \) for all \( \pi_0 \in N \). That is, \( |\pi_0(a)| > \varepsilon^{-1} \) for all \( \pi_0 \in N \), which implies \( \hat{a} \) is continuous at \( \text{cl}(\pi) \). We have seen that \( \hat{a} \) is continuous at each element of \( M_T \), hence is a continuous function. \[ \square \]

**Proposition 3.5.** The set of evaluation maps

\[ E_T = \{ \hat{a} \mid a \in S(D) \cap \mathcal{A}(T) \} \]

separates points of \( M_T \).

**Proof:** Let \( \text{cl}(\pi_1) \neq \text{cl}(\pi_2) \) in \( M_T \). Then there exists an element \( a \in S(D) \) such that \( \pi_1(a) \neq \pi_2(a) \) by Theorem 3.5. Since these are elements of \( \mathbb{R} \cup \{ \infty \} \), we may assume \( \pi_1(a) < \pi_2(a) \).
Case 1: \( \pi_2(a) \neq \infty \). Choose \( q \in \mathbb{Q} \) such that \( \pi_1(a) < q < \pi_2(a) \) and set \( b = a - q \) so that \( \pi_1(b) < 0 < \pi_2(b) \). Then \( b(1 + b^2)^{-1} \) lies in \( A(T) \) by Lemma 3.3 and clearly separates \( \pi_1 \) and \( \pi_2 \).

Case 2: \( \pi_2(a) = \infty \). Let \( n > |\pi_1(a)| \) be a natural number. Then \( 0 < \pi_1(a + n) < \pi_2(a + n) = \infty \) and so \( 0 = \pi_2((a + n)^{-1}) < \pi_1((a + n)^{-1}) < \infty \). Now apply Case 1 to \((a + n)^{-1}\) to obtain an element of \( A(T) \) separating \( \pi_1 \) and \( \pi_2 \).

**Corollary 3.6.** The space \( \mathcal{M}_T \) is Hausdorff.

**Proof:** Let \( cl(\pi_1) \neq cl(\pi_2) \) in \( \mathcal{M}_T \). Apply Proposition 3.5 to obtain \( a \in A(T) \) such that \( \pi_1(a) \neq \pi_2(a) \) in \( \mathbb{R} \). Choose two open intervals \( I_j, j = 1, 2, \) such that \( \pi_j(a) \in I_j \) and \( I_1 \cap I_2 = \emptyset \). By Theorem 3.4, we have disjoint open neighborhoods \( \hat{a}^{-1}(I_1) \) and \( \hat{a}^{-1}(I_2) \) in \( \mathcal{M}_T \) separating \( cl(\pi_1) \) and \( cl(\pi_2) \).

**Theorem 3.7.** The topology on \( \mathcal{M}_T \) is the weak topology (cf. [GJ, §3.3]) induced by \( \mathcal{E}_T \). A subbasis for the topology is given by the sets

\[
H_T(a) = \{ cl(\pi) \in \mathcal{M}_T \mid \pi(a) > 0 \} \quad (a \in A(T) \cap S(D))
\]

\[
= \{ cl(\pi) \in \mathcal{M}_T \mid \hat{a}(cl(\pi)) > 0 \} \quad (\hat{a} \in \mathcal{E}_T)
\]

**Proof:** Denote the quotient topology on \( \mathcal{M}_T \) by \( T \) and the weak topology by \( T_\mathcal{E} \). By Theorem 3.4, we know that \( T \) is a finer topology than \( T_\mathcal{E} \), so the identity map \( (\mathcal{M}_T, T) \to (\mathcal{M}_T, T_\mathcal{E}) \) is continuous. Therefore \( (\mathcal{M}_T, T_\mathcal{E}) \) is a compact space. The argument presented in the proof of Corollary 3.6 applies equally well to \( (\mathcal{M}_T, T_\mathcal{E}) \) since the continuous maps of \( \mathcal{E}_T \) separate points by Proposition 3.5. Thus \( \mathcal{M}_T \) is compact and Hausdorff in both topologies; but a bijective continuous map between two such spaces is always a homeomorphism, hence \( T = T_\mathcal{E} \). A basis for \( T_\mathcal{E} \) is given by the sets \( \hat{a}^{-1}(r, s) \) for...
for any open interval \((r, s)\) in \(\mathbb{R}\), \(a \in S(D) \cap A(T)\). As in the proof of Theorem 3.4, it suffices to assume \(r, s \in \mathbb{Q}\). But then, as \(\hat{a}^{-1}(r, s) = H_T(a-r) \cap H_T(s-a)\), the sets \(H_T(a)\) form a subbasis. 

Let \(C(M_T, \mathbb{R})\) be the ring of continuous real-valued functions on \(M_T\). We note that \(E_T\) is a subring of \(C(M_T, \mathbb{R})\) since, for \(a, b \in S(D) \cap A(T)\), we have \(\hat{a} \cdot \hat{b} = [(ab + ba)/2]^*\). Since \(E_T\) contains the constant \(1\) and separates points of \(M_T\), the Stone-Weierstrass Theorem \([GJ, \S 16.2]\) implies that \(E_T\) is dense in \(C(M_T, \mathbb{R})\).

We shall write \(A(T)^\times\) for the group of units in the ring \(A(T)\).

**Theorem 3.8.**

1. \(E_T\) is dense in \(C(M_T, \mathbb{R})\) with respect to the sup-norm.
2. For \(a \in S(D) \cap A(T)\), the function \(\hat{a}\) is positive definite iff \(q + a \in T\) for all \(q \in \mathbb{Q}^+\).
3. For \(a \in S(D) \cap A(T)\), the function \(\hat{a}\) is identically zero iff \(q \pm a \in T\) for all \(q \in \mathbb{Q}^+\) iff \(a \in m_p\) for all \(P \in X_T\).
4. \(T \cap A(T)^\times = \{a \in A(T) \cap S(D) \mid \hat{a}(M_T) > 0\}\) and \(S(D) \cap A(T)^\times = \{a \in A(T) \cap S(D) \mid 0 \notin \hat{a}(M_T)\}\).

**Proof:** (1) was proved above. For (2), let \(a \in S(D) \cap A(T)\). Assume \(q + a \in T\) for all \(q \in \mathbb{Q}^+\). Then for any \(\lambda \in M_T\), we have \(q + \hat{a}(\lambda) = q + \lambda(a) \geq 0\) for all \(q \in \mathbb{Q}^+\), and therefore \(\hat{a}(\lambda) \geq 0\). Conversely, assume \(\hat{a}\) is positive definite, \(q \in \mathbb{Q}^+\) and \(P \in X_T\). Then \(\lambda(P)(q + a) = q + \lambda(P)(a) > 0\) and hence \(q + a \in P\). This holds for all \(P \in X_T\), so \(q + a \in T = \bigcap_{P \in X_T} P\).

The first part of (3) is immediate from (2). The second part follows from the definition of \(m_p\) since \(a\) is symmetric. Statement (4) follows easily from the observation that \(A(T)^\times = \bigcap_{P \in X_T} A(P)^\times\).
We conclude this section with a look at the relationships between $\mathcal{M}_D$ and $X_D$ and their subspaces $\mathcal{M}_T$ and $X_T$.

**Proposition 3.9.** Let $A$ and $B$ be disjoint closed sets in $X_D$. The sets $\lambda(A)$ and $\lambda(B)$ are disjoint in $\mathcal{M}_D$ if there exists an element $d \in S(D)^{\times} \cap \mathcal{H}(D)$ such that $d$ is a unit in the valuation ring at any place in $\lambda(A \cup B)$, $d$ is in $P$ for all $P \in A$ and $d$ is in $-Q$ for all $Q \in B$.

**Proof:** ($\Rightarrow$) The sets $A$ and $B$ are closed and hence compact in $X_D$. Thus their images $\lambda(A)$ and $\lambda(B)$ are compact subsets of a Hausdorff space and hence are closed in $\mathcal{M}_D$. Assuming they are disjoint, Urysohn's Lemma says there exists a continuous $f : \mathcal{M}_D \to \mathbb{R}$ which is $1$ on $\lambda(A)$ and $-1$ on $\lambda(B)$. By Theorem 3.8, there exists $d \in S(D)^{\times} \cap \mathcal{H}(D)$ such that

\begin{equation}
|d(\ell(\pi)) - f(\ell(\pi))| < 1
\end{equation}

for all real $\mathbb{H}$-valued $*$-places $\pi$. Now if $P \in A$, (3.10) yields $|\lambda(P)(d) - 1| < 1$, whence $\lambda(P)(d) > 0$, $d \in P$ and $d$ is a unit in $A(P)$. And if $Q \in B$, (3.10) yields $|\lambda(Q)(d) + 1| < 1$, so that $\lambda(Q)(d) < 0$, $d \in -Q$ and $d$ is a unit in $A(Q)$.

($\Leftarrow$) If $\lambda(A) \cap \lambda(B) \neq \emptyset$, there exist $P \in A$ and $Q \in B$ such that $\lambda(P) = \lambda(Q)$. By hypothesis, there exists $d$, a unit in both $A(P)$ and $A(Q)$, with $d \in P$ and $d \in -Q$. But then $0 < \lambda(P)(d) = \lambda(Q)(d) < 0$, a contradiction. \hfill $\blacksquare$

**Proposition 3.11.** For any two preorderings $T_1$ and $T_2$ in $D$, we have $\mathcal{M}_{T_1 \cap T_2} = \mathcal{M}_{T_1} \cup \mathcal{M}_{T_2}$.

**Proof:** Every $*$-ordering containing $T_1$ or $T_2$ contains $T_1 \cap T_2$, hence $X_{T_i} \subseteq X_{T_1 \cap T_2}$ ($i = 1, 2$). Applying $\lambda$ gives $\mathcal{M}_{T_1} \cup \mathcal{M}_{T_2} \subseteq \mathcal{M}_{T_1 \cap T_2}$. Assume there exists $P \in X_{T_1 \cap T_2}$ such that $\lambda(P) \notin \mathcal{M}_{T_1} \cup \mathcal{M}_{T_2}$. By Proposition...
3.9, there exists \( d \in P \) which is negative with respect to any *-ordering in \( X_{T_1} \cup X_{T_2} \). Since any preordering equals the intersection of all *-orderings containing it, \(-d\) lies in \( \bigcap \{ P \mid P \in X_{T_1} \} \cap \bigcap \{ P \mid P \in X_{T_2} \} = T_1 \cap T_2 \subseteq P \), a contradiction. 

An important special class of preorderings are the fans. A preordering is said to be a fan if for any \( b \in S(D)^\times \) with \( b \notin -T \), we have \( 1 + b \in T \cup bT \). Their importance and several characterizations of them are shown in [C3; C4]. We shall see that one method of obtaining fans is via the wedge product introduced earlier. We write \( T_\pi \) for the preordering \( T \land P_\pi \).

**Theorem 3.12.** Let \( T \) be a preordering of \( D \) and \( c\ell(\pi) \in \mathcal{M}_T \).

1. The preordering \( T_\pi \) is a fan.
2. For any \( P \in X_T \), we have \( P \in X_{T_\pi} \) iff \( \lambda(P) = c\ell(\pi) \).
3. \( \mathcal{M}_{T_\pi} = \{ c\ell(\pi) \} \).
4. \( T_\pi = \bigcap \{ P \in X_T \mid \lambda(P) = c\ell(\pi) \} \).
5. \( T = \bigcap \{ T_\pi \mid c\ell(\pi) \in \mathcal{M}_T \} \) and no \( c\ell(\pi) \) can be omitted from the intersection.
6. \( X_T \) is the disjoint union of the sets \( X_{T_\pi} \), one \( \pi \) from each equivalence class in \( \mathcal{M}_T \).

**Proof:**

1. We know that \( T_\pi \) is fully compatible with the valuation \( v \) associated with \( \pi \) and that \( T_\pi \) pushes down to the *-ordering \( P_\pi \). By [C4, Proposition 2.9], this implies that \( T_\pi \) is a fan.

2. First assume that \( P \in X_{T_\pi} \). Then \( \lambda(P) = c\ell(\pi) \) by Lemma 3.1. Conversely, if \( \lambda(P) = c\ell(\pi) \), then \( \bar{P} = P_\pi \) in the residue *-field, so \( \pi^{-1}(P_\pi) \subseteq P^e \). Therefore \( T_\pi = \{ \sum t_i u_i \mid t_i \in T^e, u_i \in \pi S(D) \cap \pi^{-1}(P_\pi) \} \cap S(D) \subseteq T^e \cdot P^e = P^e \), and hence \( P \in X_{T_\pi} \).

3. (4) and (6) follow immediately from (2).
The equation $T = \bigcap T_\pi$ follows from (4). The remainder follows from (3) and Proposition 3.11.  

4. Places and valuations. Let $T$ be a preordering of $(D, *)$. There is a natural choice for an associated $*$-valuation $v_T$, namely the valuation associated with the valuation ring

$$A_T = \prod_{P \in X_T} A(P)$$

It is shown in [C4] that $v_T$ is fully compatible with $T$; i.e. $v_T$ is compatible with each $*$-ordering $P$ containing $T$. Furthermore, $v_T$ is the finest $*$-valuation which is fully compatible with $T$. Let $\bar{D}$ be the residue $*$-field of $v_T$ and let $\pi : D \to \bar{D} \cup \{\infty\}$ be the associated place. In this section we shall study the relationship between the $\mathbb{H}$-valued $*$-places of $D$ and those of $\bar{D}$.

**Proposition 4.1.** Let $T$ be a preordering and let $\pi : D \to \bar{D} \cup \{\infty\}$ be as above. For each $P \in X_T$ and each $\pi_0 \in \lambda(P)$, there exists a place $\pi_1 \in \lambda(\bar{P})$ such that $\pi_0 = \pi_1^+ \circ \pi$, where $\pi_1^+ : \bar{D} \cup \{\infty\} \to \mathbb{H} \cup \{\infty\}$ extends $\pi_1$ by mapping $\infty \mapsto \infty$.

**Proof:** By [C4, Proposition 2.6], we have $A(\bar{P}) = \overline{A(P)}$. Writing $m_r$ (resp., $m_p$) for the maximal ideal of $A_T$ (resp., $A_P$), we have $m_r \subseteq m_p$. Hence there exists a canonical mapping $A(\bar{P}) \to A(P)/m_p$ whose kernel is $m$, the maximal ideal of $A(\bar{P})$. The inclusion of the residue field $A(P)/m_p$ into $\mathbb{H}$ determined by $\pi_0$ thus induces an inclusion of $A(\bar{P})/m_p$ into $\mathbb{H}$, which in turn induces the desired place $\pi_1$ on $\bar{D}$.  

**Theorem 4.2.** Let $T$ be a preordering with associated valuation $v_T$ and place $\pi : D \to \bar{D} \cup \{\infty\}$. Let $\mathcal{T}$ be the pushdown preordering of $\bar{D}$. Then composition with $\pi$ induces a homeomorphism between $\mathcal{M}_T$ and $\mathcal{M}_T$.  

**Proof:** Given any \( H \)-valued \(*\)-place in an equivalence class in \( \mathcal{M}_T \), composing with \( \pi \) yields an \( H \)-valued \(*\)-place on \( D \). It is clear that this composition respects equivalence classes, giving us an injection \( \mathcal{M}_T \rightarrow \mathcal{M}_D \). By [C4, Theorem 2.2], the image lies in \( \mathcal{M}_T \) and by Proposition 4.1, the mapping is onto \( \mathcal{M}_T \). We thus have a commutative diagram

\[
\begin{array}{ccc}
X_T & \longrightarrow & X_T \\
\downarrow && \downarrow \\
\mathcal{M}_T & \longrightarrow & \mathcal{M}_T
\end{array}
\]

where the top mapping is a surjection onto a quotient space by [C4, Theorem 2.2] as are the vertical mappings (Proposition 2.7). It is then clear that the bottom mapping and its inverse are both continuous, whence it is a homeomorphism. \( \square \)

**Corollary 4.3.** \( |\mathcal{M}_T| = 1 \) iff \( \mathcal{T} \) is a \(*\)-ordering of \( \tilde{D} \).

**Proof:** If \( \mathcal{T} \) is a \(*\)-ordering, then \( |X_T| = |\mathcal{M}_T| = 1 \), from which it follows that \( |\mathcal{M}_T| = 1 \) by Theorem 4.2. Conversely, assume \( |\mathcal{M}_T| = 1 \). Again using Theorem 4.2, we have \( |\mathcal{M}_T| = 1 \), so that all valuation rings of \(*\)-orderings in \( X_T \) are equal. Therefore, for each \( P \in X_T \), we have \( A(P) = A_T \), which equals \( \tilde{D} \) by [C4, Proposition 2.6]. This says the \(*\)-orderings of \( \tilde{D} \) are archimedean orderings of the center of \( \tilde{D} \). If two of them differed, the induced \(*\)-embeddings of \( \tilde{D} \) into \( H \) would differ on a symmetric element of \( \tilde{D} \), contradicting \( |\mathcal{M}_T| = 1 \). Therefore \( \mathcal{T} \) is the only such \(*\)-ordering. \( \square \)

For any \(*\)-valuation \( \nu \) with value group \( \Gamma \), we write \( S(\Gamma) \) for the set \( \nu'(S(D)^\times) \). This is a subgroup of \( \Gamma \) if \( \nu \) is real since \( \nu(a) + \nu(b) = \nu(ab + ba) \) for \( a, b \in S(D)^\times \) [C2, Lemma 1.4].
Lemma 4.4. Let $T = P \cap Q$ for two distinct $*$-orderings $P, Q \in X_D$. Let $v_p : D^\times \to \Gamma$ be the canonical $*$-valuation associated with $A(P)$.

1. If $\lambda(P) \neq \lambda(Q)$, then $v_p(T) = S(\Gamma)$.
2. If $\lambda(P) = \lambda(Q)$, then $[S(\Gamma) : v_p(T)] = 2$.

Proof: Note first that $v_p(T)$ is a subgroup of $S(\Gamma)$, closed under addition in the value group for the same reason as $S(\Gamma)$.

1. Let $\pi_p \in \lambda(P)$, $\pi_q \in \lambda(Q)$. Since $\lambda(P) \neq \lambda(Q)$, we may apply Proposition 3.5 to obtain an element $a \in S(D) \cap A(T)$ such that $\pi_p(a) \neq \pi_q(a)$ in $R$. Relabelling and adding a rational number if necessary, we may assume $\pi_q(a) < 0 < \pi_p(a)$. Thus $a \in P, a \notin Q$ and therefore $a \notin T$. Now $T = P \cap Q$ implies $[P^e \cap \Pi S(D) : T^e \cap \Pi S(D)] = 2$ (cf. [C3, Proposition 6.5], [C4, §2]), so that the group $\Pi S(D)$ is generated by $T^e \cap \Pi S(D), -1$ and $a$. Now $v_p(a) = v_p(-1) = 0$, so $v_p(T) = v_p(\Pi S(D)) = S(\Gamma)$.

2. Since $\lambda(P) = \lambda(Q)$, we have $A(P) = A(Q)$ and therefore $v_p$ is fully compatible with $T$. Furthermore, $P = Q = T$ in the residue $*$-field $\bar{D}$ of $v_p$, so [C4, Theorem 2.5] implies

$$[S(\Gamma) : v_p(T)] = [\Pi S(D) : T^e \cap \Pi S(D)]/[\Pi S(\bar{D}) : T^e \cap \Pi S(\bar{D})] = 4/2 = 2$$

In Theorem 3.12, we have seen that for the special fans $T_\pi$, the associated subsets of $\mathcal{M}_D$ have only one element. We conclude this section by looking at the situation for an arbitrary fan.

Proposition 4.5. Let $T$ be a preordering of $(D, *)$. Then $T$ is a fan if and only if

1. $|\mathcal{M}_T| \leq 2$ and
2. for $cl(\pi_i) \in \mathcal{M}_T$, $i = 1, 2$, $T : U_{\pi_1}$ and $T : U_{\pi_2}$ are equal subgroups of $\Pi S(D)$, where we write $U_\pi$ for the multiplicative group $\Pi S(D) \cap \pi^{-1}(H^\times)$. 
PROOF: First assume that $T$ is a fan. By Theorem 4.2, we have $|\mathcal{M}_T| = |\mathcal{M}_\mathcal{T}|$, where the bar now denotes reduction to the residue $*$-field of $v_T$. But $|\mathcal{M}_\mathcal{T}| \leq |X_T|$ which is at most two by [C4, Theorem 2.13], whence (1) holds. For (2), we first note that $U_\pi$ depends only on $c\ell(\pi)$ since $\Pi S(D) \cap \pi^{-1}(H^\times) = \Pi S(D) \cap \pi^{-1}(R^\times)$; indeed, any unit in $\Pi S(D)$ can be written as the product of a symmetric unit and a unit in $\Sigma(D)$ [C4, Lemma 2.4], both of which necessarily push down into $R^\times$.

Next we show that the subset $T \cdot U_\pi$ of $\Pi S(D)$ is actually a subgroup. Since $U_\pi$ contains commutators of elements of $\Pi S(D)$, the set $(T^e \cap \Pi S(D)) \cdot U_\pi$ is closed under multiplication and contains $T \cdot U_\pi$. It will suffice to show these are equal. Let $t = s_1 \cdots s_n$ be a product of symmetric elements with $t \in T^e$. Let $u \in U_\pi$. Then $tu = (t + t^*)u'$ where $t + t^* \in T$ and $u' = (t + t^*)^{-1}tu \in U_\pi$ since $(t + t^*)^{-1}t \in \Pi S(D)$ with $\pi((t + t^*)^{-1}t) \neq 0$ [C2, Lemma 1.4].

Now let $\tilde{\pi}_i$ be the places on $\tilde{D}$ corresponding to the places $\pi_i$ as given by Proposition 4.1. We may assume $c\ell(\tilde{\pi}_1) \neq c\ell(\tilde{\pi}_2)$ since (2) is trivial otherwise; then we also have $|X_{\mathcal{T}}| = 2$, so $\tilde{T}$ is the intersection of two $*$-orderings. The place $D \to \tilde{D} \cup \{\infty\}$ induces surjections $U_{\pi_i} \to U_{\tilde{\pi}_i}$. Applying Lemma 4.4(1) to $\tilde{D}$, we have $\tilde{T} \cdot U_{\tilde{\pi}_i} = \Pi S(\tilde{D})$, $i = 1, 2$. Let $u \in U_{\tilde{\pi}_1}$. Then $\tilde{u}$ in $U_{\pi_1}$ can be written as $\tilde{u} = t\tilde{u}_2$, for some $t \in T \cap A_T$ and some $u_2 \in U_{\pi_2}$. Write $m = u - tu_2$, an element of $m_T$, the maximal ideal of $A_T$. Since $t \neq 0$, $t$ is a unit in $A_T$ and therefore $u_2 + t^{-1}m \in \pi_2^{-1}(H^\times)$. Also $u_2 + t^{-1}m = t^{-1}u \in \Pi S(D)$, so $u = t(u_2 + t^{-1}m) \in T \cdot U_{\pi_2}$. The reverse inclusion is similar, so we obtain $T \cdot U_{\pi_i} = T \cdot U_{\tilde{\pi}_i}$.

For the converse, assume (1) and (2) hold. From (1), we know $|\mathcal{M}_\mathcal{T}| \leq 2$. If $|\mathcal{M}_\mathcal{T}| = 1$, then $T$ is a fan by [C4, Proposition 2.9]. Assume as above that $\mathcal{M}_T = \{c\ell(\pi_1), c\ell(\pi_2)\}$. If each place $\tilde{\pi}_i$ has only one compatible $*$-ordering, then $T$ is the intersection of two $*$-orderings and $T$ is a fan by [C4,
Theorem 2.13]. Thus we may assume \( \pi \) has two compatible \(*\)-orderings and there exists an element \( a \in D \) which is positive in one and negative in the other. Also, we have \( A_{\pi_1} \cdot A_{\pi_2} = A_{\pi} \), which in turn equals \( D \) [C4, Proposition 2.6], so the corresponding valuations are independent. As in the commutative case, there is an approximation theorem (see, for example, [M]) which guarantees the existence of an element \( d \in D \) such that, writing \( v_1 \) for the valuation associated with \( \pi_1 \), \( v_1(d - a) > 0 \) and \( v_2(d - 1) > 0 \). But then \( d \in T \cdot U_{\pi_1} \) and \( d \notin T \cdot U_{\pi_2} \), contradicting (2), which implies these two groups must be equal. Therefore each place \( \pi \) has only one compatible \(*\)-ordering and \( T \) is a fan.

5. The holomorphy ring. As in the previous sections, we shall continue to work with the more general class of rings \( A(T) \) associated with preorderings. In Lemma 3.3, we determined a few properties of these rings. In this section, we shall find an explicit characterization of the elements of \( A(T) \) and determine some of its ring theoretic properties.

Our next theorem differs in an essential way from the standard theory (see [L, §11.4]). However the complications arise not from the fact that \( D \) may be noncommutative, but from the situation where \( * \neq \text{identity} \). This forces us to use part (2) of Lemma 3.3 rather than part (1) as the main characteristic of our rings \( A(T) \).

We follow Gräter [Gr] in generalizing the notion of Prüfer ring to the noncommutative setting. Let \( A \) be a left Ore domain with field of fractions \( D \). We say \( A \) is a Prüfer ring if for every maximal left ideal \( m \) of \( A \), the set \( S = A \setminus m \) is a left denominator set and \( S^{-1}A \) is a total subring of \( D \). Our interest is only in the case where \( A \) is invariant. Then it is easily seen that every ideal is 2-sided and we require no distinction between left and right. In this case, [Gr, Lemma 2.1] shows that for any completely prime
ideal \( \phi \), the localization \( A_\phi \) is a total subring. For further information on noncommutative localizations, see [Co1], [Co2, §1.2], [M, pp. 39–40], [P] and Proposition 5.6 below.

**Theorem 5.1.** Let \( D \) be a \(*\)-field with \(-1\) not a norm. Let \( S \) be the set of all elements of the form

\[
d(1 + dd^*)^{-1}, \quad d^*(1 + dd^*)^{-1}, \quad \text{or} \quad (1 + dd^*)^{-1}, \quad d \in D.
\]

Then any subring \( A \) of \( D \) containing \( S \) is a Prüfer domain with \( D \) as its field of fractions.

**Proof:** We must show that the localizations of \( A \) at prime ideals are total subrings of \( D \). So we may assume that \( A \) is a local ring with maximal ideal \( \mathfrak{m} \).

Let \( d \in D \) and assume \( d \notin A \). Since \( d(1 + dd^*)^{-1} \in A \), we know \( 1 + dd^* \notin A \).

But \( (1 + dd^*)^{-1} \in A \) and therefore it lies in \( \mathfrak{m} \). Then \( dd^*(1 + dd^*)^{-1} = 1 - (1 + dd^*)^{-1} \in 1 + \mathfrak{m} \), which is contained in the units of \( A \). Now \( d^*(1 + dd^*)^{-1} \) also lies in \( A \), hence so does \( d^{-1} = [d^*(1 + dd^*)^{-1}][dd^*(1 + dd^*)^{-1}]^{-1} \).

**Corollary 5.2.** For any preordering \( T \) of \((D,*)\), the ring \( A(T) \) is a Prüfer domain.

**Proof:** This follows immediately from Lemma 3.3 and Theorem 5.1.

In analogy with the way in which \( A(P) \) was defined, we might also consider the ring

\[
R = \{ d \in D \mid n - dd^* \in T, \text{ for some } n \in \mathbb{Z}^+ \}
\]

associated with the preordering \( T \).

By definition of \( R \), if \( d \in R \), then there exists a positive integer \( n \) such that \( n - dd^* \in T = \bigcap_{P \in X_T} P \). In particular, we have \( d \in A(P) \) for every
*-ordering $P$ containing $T$, and hence $d \in A(T) = \bigcap_{P \in X_T} A(P)$. Thus we see that $R \subseteq A(T)$. When $* = \text{identity}$, one has $R = A(T)$. We could show that $R$ is a Prüfer domain directly from Theorem 5.1, and we could then prove $R = A(T)$ if we knew that the maximal ideals of $R$ were $*$-closed, as then each localization of $R$ would contain some $A(P)$ for $P \in X_T$. Rather than analyzing maximal ideals, we shall prove $R = A(T)$ by proving directly that any symmetric element in $D$ which is bounded by some positive integer with respect to each $P \in X_T$ is actually uniformly bounded.

**Theorem 5.3.** For any preordering $T$, we have

$$A(T) = \{ d \in D \mid n - dd^* \in T, \text{ for some } n \in \mathbb{Z}^+ \}$$

**Proof:** Assume that $d \in A(T)$; i.e., for each $P \in X_T$, there exists an $n \in \mathbb{Z}$ such that $dd^* < n$. To show $d \in R$, we must show there exists an $n \in \mathbb{Z}$ which bounds $dd^*$ for all $P \in X_T$. Set $s = dd^*$. If $* = \text{identity}$, the result is well-known [L, Theorem 11.2]. Now $F = Z_D \cap S(D)$ is a subfield of $D$ (with identity involution) and every $P \in X_T$ restricts to some ordering $P \cap F$ with respect to which $s$ is bounded. We claim that $s$ is bounded with respect to every ordering of $F$ containing the preordering $T \cap F$ and therefore $s$ is uniformly bounded with respect to all orderings of $T$ containing $T \cap F$, and hence with respect to all $*$-orderings of $D$ containing $T$.

To prove the claim, assume that $Q \supseteq T \cap F = \bigcap_{P \in X_T} (P \cap F)$ is an ordering of $F$ which does not extend to a $*$-ordering of $D$. Note, in particular, that $Q$ is not among the orderings of $F$ in the intersection. Applying parts (5) and (6) of Theorem 3.12 to $T \cap F$, we see that some $P \cap F$ has the same associated real place as $Q$. Thus $s$ is also bounded with respect to $Q$. 

We can use the preceding theorem to obtain an explicit form for all elements of $A(T)$. 

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Corollary 5.4. $A(T) = \{nd^*(1 + dd^* + t)^{-1} \mid d \in D, t \in T, n \geq 0 \text{ in } \mathbb{Z}\}$

Proof: All elements of the given form lie in $A(T)$ by Lemma 3.3. Let $x \in A(T)$ be an arbitrary nonzero element. Since $A(T)$ is $*$-closed, we have $2x^* \in A(T)$. By Theorem 5.3, this implies there exists a positive integer $n$ such that $t_0 = n^2 - 4x^* x \in T$. Then $t_0(4x^* x)^{-1} = n^2(4x^* x)^{-1} - 1$, or equivalently,

$$x^{-1} x^{-1} n^2 / 2 = n^2 (4x^* x)^{-1} + n^2 (4x^* x)^{-1} = 1 + n^2 (4x^* x)^{-1} + t_0(4x^* x)^{-1}.$$  \hfill (5.5)

Setting $d = nx^{-1} / 2$ and $t = t_0(4x^* x)^{-1}$, this yields $nx^{-1} d^* = 1 + dd^* + t$, or $x = nd^*(1 + dd^* + t)^{-1}$, where $t = t^*$ follows from equation (5.5) so that $t \in T^* \cap S(D) = T$. □

Note that, since any $*$-valuation ring contains all elements of the form $d^* d^{-1}$, $d \in D^*$, the ring $A(T)$ also contains these elements. We next collect a few elementary facts about such rings.

Proposition 5.6. Let $A$ be a subring of $(D, *)$ containing $\{d^* d^{-1} \mid d \in D^*\}$, and let $\mathfrak{p}$ be a prime ideal of $A$. Then

1. $A$ is $*$-closed.
2. $A$ contains $[D^*, D^*]$.
3. $A$ is an invariant subring of $D$.
4. Any left (or right) ideal of $A$ is $*$-closed and 2-sided.
5. $\mathfrak{p}$ is a completely prime ideal.
6. $S = A \setminus \mathfrak{p}$ is a denominator set; i.e., $0 \notin S$, $S$ is multiplicatively closed and $S$ satisfies the Ore condition:

   for $a \in A$, $s \in S$, there exist $b \in A$, $t \in S$ such that $at = sb$. 
(7) If $A$ is Prüfer, the localization

$$A_{\mathcal{P}} = \{as^{-1} \mid a \in A, \ s \in S\}$$

is a $\ast$-valuation ring.

**Proof:** (1)-(4) are contained in [H1, 2.2].

(5) Let $x, y \in A$ with $xy \not\in \mathcal{P}$. To show that $x$ or $y$ lies in $\mathcal{P}$, it suffices to show that $(x)(y) = (xy)$ and use the fact that $\mathcal{P}$ is prime. But for any $a \in A$, $x ay = xya[a^{-1}, y^{-1}] \in (xy)$ by (2), yielding the desired equality of ideals.

(6) $\mathcal{P}$ is multiplicatively closed because $\mathcal{P}$ is completely prime. The Ore condition follows from (3); indeed, $b = s^{-1}as \in A$, so $as = sb$ yields the desired result.

(7) Since $A$ is Prüfer, $A_{\mathcal{P}}$ is a total subring of $D$. But then $A_{\mathcal{P}}$ is a $\ast$-valuation ring since it contains $A$ which contains all $d^\ast d^{-1}$ for $d \in D^\ast$. 

**Theorem 5.7.** Let $T$ be a preordering of $(D, \ast)$ and let $(E, \ast)$ be any other $\ast$-field. There exists a natural one-to-one correspondence between the set of $\ast$-places from $D$ to $E \cup \{\infty\}$ compatible with $T$ and the set $\text{Hom}(A(T), E)$ of $\ast$-ring homomorphisms from $A(T)$ into $E$.

**Proof:** Let $\pi : D \to E \cup \{\infty\}$ be a $\ast$-place compatible with $T$. Compatibility says that $\pi$ is finite on $A(P)$ for some $P \in X_T$, and hence $\pi$ is finite on $A(T)$. Therefore the restriction of $\pi$ to $A(T)$ is a $\ast$-preserving ring homomorphism $A(T) \to E$. To see that this correspondence is bijective, let $f : A(T) \to E$ be any $\ast$-homomorphism; we shall show that $f$ extends uniquely to a $\ast$-place $\pi : D \to E \cup \{\infty\}$ and that $\pi$ is compatible with $T$. Since the image of $f$ is a domain, $\mathcal{P} = \ker f$ is a completely prime ideal in $A(T)$. 


By Proposition 5.6, we can form the localization $A(T)_\nu$, a $*$-valuation ring. The usual universal property for localizations [Co2] implies that $f$ extends uniquely to a homomorphism $f_1 : A(T)_\nu \to E$; it is easily seen that $f_1$ is again a $*$-homomorphism. Now extend $f_1$ to $D$ by setting $f_1(d) = \infty$ for any $d \in D \setminus A(T)_\nu$. This is a $*$-place extending $f$.

To show compatibility with $T$, let $m = pA(T)_\nu$. By [C3, Proposition 4.5], if $T$ is not compatible with $A(T)_\nu$, then there exists $t \in T$ such that $1 + t \in m$. But $(1 + t)^{-1} \in A(T)$ by Lemma 3.3, so this cannot happen.

To prove uniqueness, let $\pi : D \to E \cup \{\infty\}$ be any other $*$-place extending $f$. Certainly $\pi$ must agree with $f_1$ on $A(T)_\nu$. Consider any $d \in D$ with $d^{-1} \in m$. Write $d^{-1} = as^{-1}$ with $a \in \nu$, $s \in A(T) \setminus \nu$. If $\pi(d) \neq \infty$, then $\pi(s) = \pi(d)\pi(a) = 0$, contradicting $s \notin \nu = \ker f$. Thus $\pi(d) = \infty$ and so $\pi$ agrees with $f_1$ on all of $D$.

When we specialize this theorem to $E = \mathbb{H}$ and $T = S(\Sigma)$, the ring $A(T)$ becomes the real holomorphy ring $\mathcal{H}(D)$. The $*$-places $D \to \mathbb{H} \cup \{\infty\}$ compatible with $S(\Sigma)$ are precisely the real $\mathbb{H}$-valued $*$-places. This set, modulo our usual equivalence relation from §2, yields $\mathcal{M}_D$. Via the previous theorem, this induces an equivalence relation on $\text{Hom}(\mathcal{H}(D), \mathbb{H})$; this is obtained by restricting the places to $\mathcal{H}(D)$ in the definitions of §2.

Since the classes of homomorphisms are actually distinguished by their actions on the symmetric elements, it is natural to consider

$$\mathcal{H}_S = \mathcal{H}(D) \cap S(D) = \{a \in \mathcal{H}(D) \mid a^* = a\}$$

A standard construction [ZSSS, p. 52] makes $\mathcal{H}_S$ into a special Jordan algebra over $\mathbb{Q}$ by giving it a new operation

$$a \odot b = (ab + ba)/2$$
to replace multiplication. In particular, \( \odot \) is commutative and distributive over addition. If we view \( \mathbb{R} \) as a Jordan algebra in which \( \odot \) coincides with the usual multiplication, restriction of homomorphisms gives

\[
\text{Hom}(\mathcal{H}(D), \mathcal{H}) \rightarrow \text{Hom}_{\mathbb{R}}(\mathcal{H}, \mathcal{H}_S, \mathbb{R}).
\]

Combining this with the correspondence of Theorem 5.6 and using our knowledge of the equivalence relation we have on places, we obtain a well-defined map

\[
\mathcal{M}_D \rightarrow \text{Hom}_{\mathbb{R}}(\mathcal{H}_S, \mathbb{R}),
\]

which is injective by Proposition 3.5. Unlike the situation when \( * = \text{identity} \), we cannot expect this to be surjective as \( \mathcal{H}_S \) generally has homomorphisms to \( \mathbb{R} \) which do not extend to homomorphisms of \( \mathcal{H}(D) \) into \( \mathcal{H} \).

The evaluation mapping of \( \S 3 \) gives a canonical (Jordan algebra) homomorphism of \( \mathcal{H}_S \) into \( C(\mathcal{M}_D, \mathbb{R}) \) whose image is \( \mathcal{E}_{\mathcal{M}_D}(\mathcal{H}_S) \) (defined in Proposition 3.5). We can compute the kernel of this mapping.

**Proposition 5.8.** With the notation above,

\[
\ker(\mathcal{H}_S \rightarrow C(\mathcal{M}_D, \mathbb{R})) = \{ a \in \mathcal{H}_S | q \neq a \in \Sigma(\mathcal{H}(D)) \text{ for all } q \in \mathbb{Q}^+ \},
\]

where \( \Sigma(\mathcal{H}(D)) = \{ \sum d_id_i^*c_i | d_i \in \mathcal{H}(D), \quad c_i \in [D^\times, S(D)^\times] \} \).

**Proof:** We first note that \( \Sigma(\mathcal{H}(D)) = \mathcal{H}(D) \cap \Sigma(D) \). Assume that \( h = \sum d_id_i^*c_i \quad (d_i \in D, \quad c_i \in [D^\times, S(D)^\times]) \) is an arbitrary element of \( \mathcal{H}(D) \cap \Sigma(D) \). For any \( P \in X_D \), we have \( v_p(h) \geq 2 \min v_p(d_i) \). Assume the minimum occurs for \( i = 1 \) and \( v_p(h) > 2v_p(d_1) \). Then

\[
(d_1d_1^*)^{-1}h = c_1 + \sum_{i>1} (d_1d_1^*)^{-1}d_id_i^*c_i
\]

\[
= c_1 + \sum_{i>1} (d_1^{-1}*d_1)(d_1^{-1}*d_1)^*c_i
\]

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where $c_i = [d_i(d_i^*)^{-1}]c_i \in \{D^\times, S(D)^\times\}$. Reducing to the residue field of $v_p$ yields $0 = 1 + \sum_{i>1} a_i a_i^*$ for some elements $a_i \in \mathcal{H}$, a contradiction. Therefore $0 \leq v_p(h) = 2 \min(v_p(d_i))$ and so each $d_i$ lies in $\mathcal{H}(D)$.

Now assume $a \in \mathcal{H}_S$ with $a = 0$. That is, $\lambda(P)(a) = 0$ for each $P \in X_D$, or equivalently, $0 < a^2 < q^2$ (with respect to $P$) for all $q \in \mathbb{Q}^+$. This implies $q \pm a \in \Sigma(D) \cap \mathcal{H}(D) = \Sigma(\mathcal{H}(D))$ as desired.

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