Fields Maximal with Respect to a Set of Orderings

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1. Introduction and Notation

In this paper, we shall study certain natural generalizations of the concept of real closure, replacing one ordering with a set of orderings. Following [C1], we make the following definitions.

Definition 1.1. A formally real field is order closed if it has no algebraic extension to which all orderings extend uniquely. An order closure of a formally real field $F$ is an order closed field to which all orderings of $F$ extend uniquely.

Using Zorn's lemma one can show that order closures always exist [C1, Sect. 4]. This is the first systematic treatment of such fields, though they have also been used in [C2]. A major problem in dealing with order closed fields is that it is not clear whether or not they have algebraic extensions to which all orderings extend (necessarily nonuniquely). This leads us to make the following definition.

Definition 1.2. A formally real field is strongly order closed (SOC) if it has no algebraic extension to which all orderings extend. A strong order closure of a field $F$ is a SOC field to which all orderings of $F$ extend uniquely.

It is clear from this definition that a SOC field is order closed. On the other hand, it is not clear that strong order closures always exist. One of the major questions addressed in this paper is whether every order closed field is SOC. This remains open in general, but in Section 4 we are able to establish the equivalence of the two concepts for two large classes of fields: those for which every odd degree extension satisfies the strong
approximation property SAP and those for which there are only finitely many places into the real numbers.

Two related concepts of closure have proven to be very important in studying ordered fields and quadratic forms, namely that of pythagorean closure \([L1]\) and Galois order closure (maximal normal extension to which all orderings extend \([C4, G]\)). These concepts play a prominent part in Section 2 where we discuss general algebraic results concerning order closed and SOC fields. In \([G]\) it is shown that a field \(F\) is Galois order closed if and only if it equals \(F^*\), the intersection of all real closures of \(F\) inside some fixed algebraic closure. In \([C4]\) we showed that \(F = F^*\) if and only if every polynomial over \(F\) which splits in each real closure of \(F\) also splits in \(F\). In Theorem 2.1 we obtain a similar characterization of SOC fields; namely, the field \(F\) is SOC if and only if every irreducible polynomial over \(F\) which has a root in each real closure of \(F\) has a root in \(F\). Another characterization is that the field is pythagorean and every minimal extension is quadratic. Also in Section 2 we establish a weak version of Rolle's theorem for SOC fields: if a polynomial has two roots in the field, then its derivative is reducible or of degree one.

Section 3 is devoted to studying the real valuations of order closed and SOC fields. No general characterization is obtained, but the use of valuation theoretic techniques is essential to the proof in Section 4 that order closed fields are often (if not always) strongly order closed. The proof hinges on a study of the behavior of orderings under odd degree extensions of pythagorean fields. Given a formally real field \(F\), we shall write \(X_F\) for the topological space of orderings of the field where the topology is determined by the Harrison subbasis, consisting of all sets of the form

\[ X_F(c) = \{ P \in X_F \mid c \in P \}, \]

where \(c \in F^* = F - \{0\}\). Given a formally real extension field \(K\) of \(F\), there is an induced continuous mapping \(X_K \rightarrow X_F\) defined by restriction of orderings \([C1, L2]\). The main theorem of Section 4 is based on Theorem 4.3 which shows that if \(F\) is a pythagorean field with only finitely many places into \(\mathbb{R}\), then any odd degree extension \(K\) of \(F\) contains an element \(c\) such that the restriction mapping \(X_K(c) \rightarrow X_F\) is a homeomorphism.

One can avoid these problems with odd degree extensions by restricting attention to 2-extensions. A quadratically order closed field is simply a pythagorean field. A quadratic order closure of a field (with all orderings extending uniquely) always exists by Zorn's lemma as do order closures. While this eliminates the need for a corresponding "strong" property, it suffers from nonuniqueness as do order closures. For example the two
orderings of the field $\mathbb{Q}(\sqrt{2})$ each extend uniquely to each of the extension fields $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{6})$, where we use $F(2^{n}\sqrt{c})$ to mean $F(c^{1/2^n}, n = 1, 2, 3, \ldots)$. Thus order closures of $\mathbb{Q}(\sqrt{2})$ exist containing either of these fields, but one contains a fourth root of 3 and the other does not since $\sqrt{3}$ changes sign in the two orderings.

The final section of the paper looks at the related quadratic form theory. Given a field $F$, we shall write $W(F)$ for its Witt ring of equivalence classes of quadratic forms $\mathbb{Q}$. Modulo its nilradical, $\text{Nil} W(F)$, which equals the torsion subgroup, $W_1(F)$, we obtain the reduced Witt ring, $W_{\text{red}}(F)$. Let $K$ be an order closure of $F$. We show that the kernel of the induced homomorphism $W(F) \to W(K)$ is $W_1(F)$. For rational function fields we are able to obtain some information on the cokernel. This leads to an interesting generalization of the characterizations of fields $F$ for which $F(x)$ satisfies SAP (hereditarily euclidean fields) [C3, P]. Let $L$ be a strong order closure of $F$ (and hence real closed if $F$ has only one ordering). We give several equivalent conditions for the induced homomorphism $W_{\text{red}}(F(x)) \to W_{\text{red}}(L(x))$ to be an isomorphism, which restrict to characterizations of hereditarily euclidean fields when $F$ has only one ordering. In particular, this is equivalent to the property that every irreducible polynomial over $F$ with a root in $L$ has a unique root in each real closure, and to the property that for each field $K$ between $F$ and $L$, restriction of orderings $X \to X_F$ is a homeomorphism (hereditary unique extension of orderings).

2. General Properties of Order Closed Fields

We begin this section with characterizations of strongly order closed fields in terms of polynomials and extension fields.

Theorem 2.1. Let $F$ be a formally real field. The following conditions are equivalent:

(a) The field $F$ is strongly order closed.

(b) If an irreducible polynomial $f$ in $F[x]$ has a root in every real closure of $F$, then it has a root in $F$ (and hence has degree 1).

(c) The field $F$ is pythagorean and every polynomial of odd degree over $F$ has a root in $F$.

(d) The field $F$ is pythagorean and every minimal extension of $F$ is quadratic.

Proof. (a) $\Rightarrow$ (b). Let $f$ be an irreducible polynomial over $F$ with a root in every real closure of $F$. Then the extension field $F = K[x]/(f)$ can
be embedded in every real closure, so every ordering of $F$ extends to $K$. Since $F$ is strongly order closed, this extension must be trivial, and so $f$ is linear.

(b) $\Rightarrow$ (c). Given any sum of squares $t$ in $F$, the polynomial $x^2 - t$ splits in every real closure of $F$ and hence in $F$. Thus every sum of squares in $F$ has a square root in $F$ and so $F$ is pythagorean. Every polynomial over $F$ of odd degree has a root in every real closure so it also has a root in $F$ by (b).

(c) $\Rightarrow$ (d). Condition (c) implies that $F$ has no odd degree extensions. Now if $E$ is some minimal extension of $F$, say contained in a finite Galois extension $K$ of $F$, then the Galois group $G = \text{Gal}(K/F)$ has 2-power order [K, Theorem 57, p. 67]. By the theory of 2-groups, $G$ has a subgroup $H$ of index 2 containing $\text{Gal}(K/E)$. Minimality of $E$ implies that it must be the fixed field of $H$ and so $E$ is a quadratic extension.

(d) $\Rightarrow$ (a). By the Artin–Schreier theory, the elements of $F$ which are positive in all orderings are precisely the sums of squares in $F$. Since $F$ is pythagorean, these are already squares in $F$. Thus any minimal extension of $F$, being a quadratic extension, will kill at least one ordering of $F$. It follows that $F$ is strongly order closed.

In the next corollary we see that this theorem gives an interesting connection between differentiation and irreducibility for SOC fields. One might think of it as a weak version of Rolle's theorem, giving reducibility rather than a root.

**Corollary 2.2.** Let $F$ be a SOC field and let $f$ be a polynomial over $F$ of degree at least 3. If its derivative $f'$ is irreducible, then the degree of $f$ is odd and $f$ is bijective as a function from $F$ to itself. In particular, if the polynomial $f$ has two roots in $F$, then $f'$ is reducible.

*Proof.* If the degree of $f'$ is odd (and hence at least 3 by hypothesis), then $f'$ is an irreducible polynomial over $F$ but has a root in $F$ by Theorem 2.1(d), a contradiction. Thus the degree of $f$ is odd and therefore it has a root in $F$. If $f$ has two roots in $F$, then $f'$ has a root in every real closure of $F$ by Rolle's theorem, contradicting Theorem 2.1(b) since $f'$ is irreducible of degree at least 2. Similarly, for any element $c$ in $F$, the polynomial $f - c$ has a unique root in $F$. That is, the polynomial $f$ induces a bijective function from $F$ to itself.

With restrictions on the number of orderings of $F$, strongly order closed fields can be related to other field-theoretic concepts. A field $F$ is said to be superpythagorean if it is pythagorean and every subgroup of $F'$ of index 2 containing $F^2$ and excluding $-1$ is an ordering of $F$ [Br1]. A field is said to be $n$-maximal if it is a subfield of an algebraically closed field which is
maximal with respect to not containing a specific set of \( n \) elements and is not \((n-1)\)-maximal. Proposition 2.3 (below) originally appeared as the equivalence of (1) and (4) in [EV, Theorem 2.1]. Our proof makes use of results on \( n \)-maximal fields proven in [BCP].

**Proposition 2.3.** Assume that \( F \) is a field with exactly two orderings. Then \( F \) is an SOC field iff it is 3-maximal.

**Proof.** By [BCP, Theorem 6.1 and Proposition 6.5], the field is 3-maximal iff it is superpythagorean and all minimal extensions are quadratic. By Theorem 2.1, this is equivalent to being SOC and superpythagorean. But SOC fields are pythagorean and pythagorean fields with two orderings are always superpythagorean [Br1]. Hence 3-maximal is equivalent to SOC in this case.

**Example 2.4.** An order closure of \( \mathbb{Q}(\sqrt{2}) \) is any maximal subfield of the algebraic numbers \( \mathbb{Q} \) missing the set \( \{\sqrt{2}, i, i\sqrt{2}\} \) to which both orderings of \( \mathbb{Q}(\sqrt{2}) \) extend.

Proposition 2.3 can be generalized, but as its proof indicates, one must restrict the number of real places on \( F \); we do this by requiring that \( F \) be superpythagorean.

**Proposition 2.5.** Assume that \( F \) has exactly \( n \) orderings. Then \( F \) is \((2n - 1)\)-maximal iff it is SOC and superpythagorean.

**Proof.** Apply Theorem 2.1(d) and [BCP, Theorem 6.1 and Proposition 6.5].

**Theorem 2.6.** Assume that \( F \) is a SOC field. Then

(a) Every pythagorean algebraic extension of \( F \) is SOC.

(b) Every euclidean closure of \( F \) is real closed.

**Proof.** Let \( K \) be a pythagorean algebraic extension of \( F \) and let \( L \) be an odd degree extension of \( K \). Let \( L' \) be the normal closure of \( L \) over \( F \) with (profinite) Galois group \( G \). Let \( H \) be a 2-Sylow subgroup of \( G \) [R, Proposition 5.2] and let \( M \) be the fixed field of \( H \) [R, Theorem 1.7]. Then \([M:F] = [G:H]\) is an odd supernatural number [R]. Since \( F \) is SOC, we have \( M = F \) and \( H = G \), contradicting the assumption that \( L \) is an odd degree extension of \( K \). It follows that \( K \) has no odd degree extensions and hence is SOC. Statement (b) follows immediately from (a) since a euclidean field is a pythagorean field with one ordering and the order closure of a euclidean field is its real closure.
The remainder of this section is devoted to obtaining a few results about order closed fields which are not necessarily SOC.

**Lemma 2.7.** Let \( F \) be a formally real field and let \( K \) be a finite Galois extension of \( F \) such that the restriction mapping \( X_K \to X_F \) is surjective. Then there exists an algebraic extension \( L \) of \( K \) such that the restriction \( X_L \to X_F \) is bijective.

**Proof.** By the primitive element theorem we may write \( K = F(\theta) \). Let \( \theta = \theta_1, ..., \theta_n \) be the roots of the minimal polynomial of \( \theta \) over \( F \). Since \( K \) is a normal extension, all \( \theta_i \) lie in \( K \). Set \( L = K(\sqrt[n]{\theta - \theta_i}, \ i = 2, ..., n) \). The orderings of \( L \) are precisely those of \( K \) in which \( \theta \) is the largest of the roots \[C1, Lemma 1\]. Given any ordering of \( F \), it extends to \( K \) by hypothesis. Let \( R \) be a real closure with respect to the given ordering. The number of extensions of the ordering to \( L \) equals the number of embeddings of \( L \) into \( R \) which restrict to the identity on \( F \) by \[P, Corollary 3.12\]. Any such embedding is determined on \( K \) by the image of \( \theta \). This embedding extends (necessarily uniquely) to \( L \) iff \( \theta \) maps to the largest of the roots in \( R \). It is always possible to make \( \theta \) the largest root because \( \text{Gal}(K/F) \) acts transitively on the \( \theta_i \).

**Proposition 2.8.** If \( K \) is order closed, then \( K \) is Galois order closed. In particular, the field \( K \) is pythagorean.

**Proof.** By the preceding lemma, any finite normal extension of \( K \) to which all orderings extend has an algebraic extension to which all orderings extend uniquely. Therefore \( K \) has no normal extension to which all orderings extend, and hence \( K \) is Galois order closed.

**Theorem 2.9.** Let \( F \) be a formally real field. The intersection of all order closures of \( F \) (inside a fixed algebraic closure \( \overline{F} \)) is \( F^* \), the Galois order closure of \( F \).

**Proof.** Let \( L \) denote the intersection of all order closures of \( F \). Any order closure \( K \) of \( F \) is an intersection of some real closures of \( F \) since by Proposition 2.8, \( K = K^* \) which is the intersection of all real closures of \( K \) \[C4\]. Write \( K = \bigcap R_i \), where the fields \( R_i \) are real closed subfields of \( \overline{F} \). Thus \( L \) is an intersection of real closures of \( F \). To show that \( L = F^* \), we must show that all real closures of \( F \) occur in the intersection. Let \( R \) be any real closure of \( F \) in \( \overline{F} \). We show that \( R \) occurs in an intersection giving one of the order closures of \( F \). Since all orderings of \( F \) extend uniquely to \( K \), some \( R_i \) is a real closure with respect to the same ordering of \( F \) as \( R \). Hence there is a unique isomorphism from \( R_i \) onto \( R \). Extend this isomorphism to
an automorphism $\sigma$ of $\mathbb{F}$. Then $\sigma(K) = \bigcap \sigma(R_i)$ is an order closure of $F$ with $R$ as one of its real closures.

From Theorem 2.1 and Proposition 2.8, we see that an order closed field is strongly order closed if and only if it has no proper odd degree extensions. In general, we can at least show that order closed fields are closed under taking odd roots.

**Proposition 2.10.** Let $K$ be an order closed field. For any odd integer $n$ and any nonzero element $c$ in $K$, the element $c$ has an $n$th root in $K$.

**Proof.** If $c$ does not have an $n$th root in $K$, then $K(c^{1/n})$ is a proper extension of $K$ to which all orderings extend uniquely because the polynomial $x^n - c$ has a unique root in each real closure of $K$. Since $K$ is order closed, this is impossible.

3. **Valuation Theory for Order Closed Fields**

In this section we establish a few facts concerning the valuation theory of order closed and SOC fields. We end with some conditions under which we can say that an order closed field is SOC. This will be pursued in much greater detail in the next section. For general facts from valuation theory the reader is referred to [E]. For connections with orderings, two good references are [L2, P]. An ordering $P$ is said to be compatible with a valuation $v$ if $0 < a < b$ (with respect to $P$) implies $v(a) > v(b)$ in the value group $\Gamma$. Every ordering of a field is compatible with some valuation with formally real residue class field [L2, Theorem 2.6]. We shall call such a valuation a real valuation. If a valuation is 2-henselian (i.e., Hensel's lemma holds for quadratic extensions), then all orderings of the field are compatible with it [L2, Theorem 3.16].

**Theorem 3.1.** Let $K$ be a field with real valuation $v$. Each of the following statements implies the next.

(a) The field $K$ is SOC.

(b) The residue field $k$ is SOC and the value group $\Gamma_v$ is odd divisible (i.e., $n\Gamma_v = \Gamma_v$ for all odd integers $n$).

(c) The henselization $\bar{K}$ with respect to $v$ is SOC.

**Proof.** (a) $\Rightarrow$ (b). Any sum of squares in $k$ can be lifted to $K$. Since $K$ is pythagorean, the sum is a square in $K$ and hence the image in $k$ is a square. Therefore $k$ is pythagorean. Assume $f$ is a monic irreducible polynomial over $k$ of odd degree at least three with lifting $f'$ in $K[x]$. Then
$f$ is irreducible over $K$, contradicting Theorem 2.1(b). Thus $K$ is SOC by Theorem 2.1(c). It follows from Proposition 2.10 that the value group is odd divisible.

(b) $\Rightarrow$ (c). The henselization is easily seen to be pythagorean by Hensel's lemma since the residue field is pythagorean. The field $\tilde{K}$ can have no proper odd degree extension since the degree of such an extension is the product of the ramification index and the residue degree; and (b) implies neither of these can be odd. Thus $\tilde{K}$ is SOC by Theorem 2.1(c).

The theorem above shows that a field which is henselian with respect to a real valuation is SOC iff its residue field is SOC and its value group is odd divisible. One-half of this can also be proved for order closed fields.

**PROPOSITION 3.2.** Let $K$ be an order closed field with a real henselian valuation $v$. Then the residue field $k$ is order closed and the value group $\Gamma$ is odd divisible.

**Proof.** It follows from Proposition 2.10 that the value group is odd divisible. If $k$ is not order closed, then there exists an algebraic extension $F$ such that every ordering of $k$ extends uniquely to $F$. Let $L$ be an extension of $K$ with the same value group $\Gamma$ and with residue field $F$. (Such a field exists by [E, Theorem 7.1].) Since $K$ is henselian, the valuation extends uniquely to $L$. All orderings of $K$ (respectively, $L$) are compatible with the valuation $v$ (respectively, the unique extension of $v$) since $K$ (respectively, $L$) is henselian [L2, Theorem 3.16]. It follows that, since $X_K$ is bijective with $\text{Hom}(\Gamma/2\Gamma, \mathbb{Z}_2) \times X_k$ and $X_L$ is bijective with $\text{Hom}(\Gamma/2\Gamma, \mathbb{Z}_2) \times X_F$ [L2, Corollary 3.11], the restriction mapping $X_L \to X_K$ must be a homeomorphism. Since $K$ is order closed, we must have $L = K$ and therefore $k$ is also order closed.

These last two results fall far short of a valuation theoretic characterization of order closed fields. One thing they do lead to is that, under the restriction that there is a valuation with respect to which all the orderings are compatible (for example, if there is a real 2-henselian valuation), an order closed field is SOC if and only if its residue class field is.

**PROPOSITION 3.3.** Let $K$ be an order closed field for which all orderings are compatible with a single valuation $v$. If the residue class field is SOC, then so is $K$.

**Proof.** The henselization of $K$ with respect to $v$ is an algebraic extension of $K$ to which each ordering extends uniquely [L2, Corollary 3.22]. Since $K$ is order closed, this means it must already be henselian. The value group is odd divisible by Proposition 3.2, so the result follows from Theorem 3.1.
Corollary 3.4. A Laurent series field of the form \( K((t))(t^{1/n}, n \text{ odd}) \) is SOC iff \( K \) is SOC.

4. When Is an Order Closed Field SOC?

This section is devoted to showing that order closed fields are strongly order closed for two large classes of fields including all fields algebraic over \( \mathbb{Q} \) or \( \mathbb{R}(x) \) and all fields with only finitely many orderings. More generally, this latter class of fields will be included in our theorems for fields with a finite number of places into the real numbers \( \mathbb{R} \). This condition is equivalent to the concept of "finite chain length in spaces of orderings" as introduced by Marshall [M; L2, Chap. 8]. The condition depends only on the reduced Witt ring of the field, an invariant of the quadratic form structure. Our primary interest is in pythagorean fields (cf. Proposition 2.8) where this is the same as the (nonreduced) Witt ring by a theorem of Pfister [L1, Chap. VIII, Sect. 4]. Our proofs will be based heavily on valuation theory.

The main theorem on order closures follows easily from Theorem 4.3 regarding odd degree extensions of pythagorean fields. A field \( K \) is said to satisfy the strong approximation property (SAP) if any clopen subset of \( X_K \) can be written as \( X_K(c) \) for some nonzero element \( c \) in \( K \). See [L2, Chap. 17] for a summary of known equivalent conditions.

Proposition 4.1. Let \( K \) be any field and let \( L \) be an odd degree extension of \( K \) which satisfies SAP. Then there exists an element \( c \) in \( L \) such that the restriction mapping \( X_L(c) \rightarrow X_K \) is a homeomorphism.

Proof. By the definition of SAP, it suffices to know that \( X_L \) contains a clopen subset mapping homeomorphically onto \( X_K \). Since the extension has odd degree, we know that \( X_L \rightarrow X_K \) is surjective. The existence of an appropriate clopen subset then follows from [Bo, Theorem 6.9].

Corollary 4.2. Let \( K \) be a field with a unique ordering and let \( L \) be an odd degree extension of \( K \). Then there exists an element \( c \) in \( L \) such that the restriction mapping \( X_L(c) \rightarrow X_K \) is a homeomorphism.

Proof. The field \( L \) satisfies SAP by [P, Corollary 9.2].

Theorem 4.3. Let \( K \) be a pythagorean field with only finitely many places into the real numbers. Let \( L = K(\alpha) \) be an odd degree extension of \( K \). Then there exists an element \( c \) in \( L \) such that the restriction mapping \( X_L(c) \rightarrow X_K \) is a homeomorphism.
Proof. The theorem will be proved by induction on the number of places of $K$ into the real numbers. This is made possible by [C5, Theorem 2.1] which shows that all reduced Witt rings can be constructed by iteratively performing one of two processes (corresponding to the two cases below) and related to the valuation theory of the field. This theorem was subsequently proved in an abstract form in [M]. The form of the theorem which we shall use here is a constructive version for pythagorean fields due to Jacob [J].

When $K$ has only one place into $\mathbb{R}$, it is a superpythagorean field and has a 2-henselian valuation whose residue class field has a unique ordering [Bri]. By Corollary 4.2 the theorem holds for the residue field. By [J, Sect. 3] it suffices to prove the theorem in the two cases stated below, the first of which includes the situation where $K$ has a unique place into $\mathbb{R}$.

Case 1. We assume the field $K$ has a 2-henselian valuation $v$ with formally real residue field $k$ satisfying the theorem for all odd degree extensions.

Let $\tilde{K}$ be a henselization of $K$ with respect to $v$. Replacing $L$ by an isomorphic field if necessary, we write $L = K(\alpha)$, where $\alpha$ also has odd degree over $\tilde{K}$. (Choose a root of an odd degree factor over $\tilde{K}$ of the minimal polynomial of $\alpha$ over $K$.) Now $\tilde{K}(\alpha)$ is henselian with respect to the unique extension of $v$, and it contains $L$. Thus $\tilde{K}(\alpha)$ contains a henselization of $L$ of $L$ with respect to some extension $v_1$ of $v$ to $L$ [E, p. 131]. Let $v_2, \dotsc, v_r$ be the remaining extensions of $v$ to $L$. Now $L, v_1$ is a henselization of $K, v$, hence it contains $\tilde{K}$, the unique henselization of $K, v$ in $\tilde{K}(\alpha)$ [E, Theorem 17.11]. It follows that $L = \tilde{K}(\alpha)$. We now have the following commutative diagram

$$
\begin{array}{ccc}
\tilde{K} & \rightarrow & L = \tilde{K}(\alpha) \\
\uparrow & & \uparrow \\
K & \rightarrow & L = K(\alpha)
\end{array}
$$

Let $k'$ be the residue field of $v_1$ on $L$, an odd degree extension field of $k$. By hypothesis there exists an element $\bar{c}$ in $k'$ such that $X_k(\bar{c}) \rightarrow X_k$ is a homeomorphism. Let $c' \in L$ be any element mapping to $\bar{c}$. Since [$\tilde{L}:\tilde{K}$] is odd, the value group extension is odd and there is an induced isomorphism between the value groups modulo squares. Since the orderings of a valued field compatible with the valuation are determined by the value group modulo squares and the orderings of the residue field [L2, Chap. 3], the restriction mapping gives a homeomorphism of $X_L(c')$ onto $X_K$. Furthermore, we also have a homeomorphism $X_K \rightarrow X_K$ since all orderings of $K$ are compatible with $v$ [L2, Theorem 3.16]. Since $c'$ is a unit with respect to
the approximation theorem of [Br2, Theorem 2.1(B)] gives us an element $c \in L$ close to $c'$ with respect to $v_1$ and close to $-1$ with respect to $v_2, ..., v_r$. In particular, the element $c$ is positive only in orderings compatible with $v_1$. We thus obtain a commutative diagram induced by restriction of orderings in the diagram above

$$
\begin{array}{ccc}
X_K & \hookrightarrow & X_L(c) \\
\downarrow & & \downarrow \\
X_K & \hookrightarrow & X_L
\end{array}
$$

where the top and left sides are homeomorphisms and the right side mapping is onto $X_L(c)$. It follows that the restriction $X_L(c) \to X_K$ is a homeomorphism.

**Case 2.** We assume the field $K = \bigcap_{i=1}^n K_i$, where each extension field $K_i$ is either a 2-henselization of $K$ with respect to a valuation $v_i$ or an euclidean closure of $K$. The space of orderings $X_K$ is the disjoint union $\bigcup X_i$, where $X_i$ is the homeomorphic image of $X_{K_i}$, consisting of all orderings of $K$ compatible with $v_i$ if $K_i$ is a 2-henselization and consisting of a single ordering if $K_i$ is an euclidean closure. Furthermore, the valuations are "independent modulo squares"; that is, given the valuation rings $A_i, A_j, i \neq j$, we have $K'/(A_i, A_j; K') = 1$ [C5; J, Theorem 3]. We assume inductively that the theorem holds for odd degree extensions of each field $K_i$. (Each field falls under either Case 1 or Corollary 4.2).

Set $L_i = K_i(\alpha), i = 1, ..., n$. Since each $K_i$ is obtained from $K$ by successive quadratic extensions, we have $L_i$ of odd degree over $K_i$ and the induction hypothesis says we can find elements $c_i$ in $L_i$ such that $X_{L_i}(c_i)$ maps homeomorphically onto $X_{K_i}$. As in Case 1, the set $X_{L_i}(c_i)$ contains only orderings compatible with a single extension of $v_i$. The residue field of $L_i$ with respect to this valuation is the same as the residue field of $L$ with respect to an extension $v_i'$ of $v_i$. Thus (as in Case 1), we may assume that all $c_i$ lie in $L$. The independence of the $v_i$ modulo squares is sufficient to allow us to apply an approximation theorem [Br2, Theorem 2.1(A)] to the appropriate extensions $v_i'$ including trivial valuations for the euclidean fields $K_i$. (Note that since $L$ has odd degree over $K$, the value groups are unchanged modulo squares in extending to $L$.) Thus we obtain $c$ in $L$ close to $c$, with respect to each $v_i'$. It follows that $X_{L_i}(c)$ maps homeomorphically onto $X_i$ for each $i$, and $X_L(c)$ maps homeomorphically onto $X_K$.

**Remark 4.4.** The theorem above leaves two interesting open questions. The first is the question of which fields $K$ satisfy the conclusions of the theorem. We know of no fields for which the theorem fails. The second question is whether the theorem can be generalized to the abstract spaces.
of orderings of Marshall [M] or even those of finite chain length. In this case, one can ask whether, given a surjection of spaces of orderings \((X_1, G_1) \to (X_2, G_2)\), there exists a subspace \((X_1(c), G_1/d)\) of \((X_1, G_1)\) such that the induced map gives a homeomorphism of \(X_1(c)\) onto \(X_2\). Indeed, there is even the question of existence of an appropriate clopen set in the abstract case, whether or not it can be represented as an \(X(c)\). The proofs of Bos [Bo] giving the existence of clopen subsets are heavily dependent on finite field extensions as are our proofs.

The result of Theorem 4.3 can be restated ring theoretically for the associated Witt rings. By Springer's theorem [L1, Theorem VIII.2.3], an extension \(L\) over \(K\) of odd degree induces an injection of Witt rings \(W(K) \to W(L)\). The extension is integral (in fact, all Witt rings are integral over the image of \(\mathbb{Z}\) inside them) so the going-up theorem implies that every prime ideal of \(W(K)\) has a prime ideal of \(W(L)\) lying over it. The minimal prime ideals correspond bijectively with the orderings in such a way that \(c\) is positive iff the form \(\langle 1, -c \rangle\) lies in the corresponding prime ideal [L1, Theorem VIII.5.3]. The maximal ideals consist of the augmentation ideal and, for each minimal prime ideal \(P\) and odd prime number \(p\), the ideal generated by \(P\) and \(p\mathbb{Z}\) [L1, Theorem VIII.5.4]. Translating Theorem 4.3. to this language and noting that the maximal ideals are taken care of when the minimal ones are, we obtain our next theorem.

**Theorem 4.5.** Under the hypotheses of Theorem 4.3, there exists an element \(c\) in \(L\) such that each prime ideal of \(W(K)\) extends uniquely to a prime ideal of \(W(L)\) containing the form \(\langle 1, -c \rangle\).

We are finally in a position to obtain our main theorem for this section.

**Theorem 4.6.** Let \(K\) be an order closed field such that either

(i) every odd degree extension of \(K\) satisfies SAP or

(ii) the field \(K\) has only finitely many places into the real numbers.

Then \(K\) is strongly order closed.

**Proof.** The field \(K\) is pythagorean by Proposition 2.7. Assume that \(K\) has an extension \(L\) which is of odd degree over \(K\). By Theorem 4.3 or Proposition 4.1, there exists an element \(c\) in \(L\) with the property that the restriction mapping \(X_L(c) \to X_K\) is a homeomorphism. The extension \(L(\sqrt[2n]{-c})\) has its space of orderings mapping homeomorphically onto \(X_K\) [C1, Lemma 1]. This contradicts the hypothesis that \(K\) is order closed. Thus \(K\) has no extension of odd degree and so every polynomial of odd degree over \(K\) has a root in \(K\). It follows that \(K\) is SOC by Theorem 2.1(c).
Remark 4.7. Examples of fields satisfying (i) above are

(a) Formally real fields for which every real place has a 2-divisible value group [P, Corollary 9.2].

(b) Formally real fields algebraic over a field with a unique ordering (these are included in (a)).

(c) Fields with only archimedean orderings (these are included in (a)).

(d) Formally real algebraic extensions of a rational function field $F(x)$ where the field $F$ is hereditarily euclidean [P, Theorem 9.4].

5. QUADRATIC FORMS

In this section we shall take a look at the relationship between the Witt ring of a field and the Witt ring of its order closure. At the end of the section we specialize to rational function fields. We begin by stating a few basic facts about Witt rings [L2, Section 1]. In general, for a formally real field $F$, the reduced Witt ring $W_{\text{red}}(F)$ can be viewed as a subring of $C(X_F, \mathbb{Z})$, the ring of continuous functions from the space of orderings to the ring of integers with the discrete topology. With this interpretation, $W_{\text{red}}(F)$ is generated by 1 and elements of the form $2^{x_0}$, where $x_0$ is the characteristic function of the set $U$. (This corresponds to the form $\langle 1, c \rangle$ in the Witt ring.)

**Theorem 5.1.** Let $K$ be an order closure of $F$ and let $\phi : W(F) \to W(K)$ be the Witt ring homomorphism induced by the inclusion of $F$ in $K$. Then the sequence

$$0 \to W_1(F) \to W(F) \to W(K)$$

is exact.

**Proof.** Since $K$ is a pythagorean field, its Witt ring is torsion free and therefore $W_1(F)$ is contained in the kernel of $\phi$ [L1, Theorem VIII.3.3]. Since $K$ is an order closure of $F$, the restriction mapping $X_K \to X_F$ is a homeomorphism. Thus we have a commutative diagram

$$W_{\text{red}}(F) \longrightarrow W(K)$$

where the vertical mappings are injections. It follows that $W_{\text{red}}(F) \to W(K)$ is an injection, showing that the desired sequence is exact.
Computation of the cokernel of $\phi: W(F) \to W(K)$ is generally a much harder problem. Our next example shows that it is not determined by $W(F)$ or even by $F$, but rather by the particular choice of order closure $K$. Prior to exhibiting this example, we describe an extreme case in which the structure is understood. One of the many known equivalent conditions for the field $K$ to satisfy SAP (cf. Sect. 4) is that its reduced Witt ring be as large as possible, namely the subring $\mathbb{Z} + C(X_K, 2\mathbb{Z})$ of $C(X_K, \mathbb{Z})$. In this case the cokernel is known (at least in principal) because there is an explicit description of the elements of $C(X_F, \mathbb{Z})$ which lie in the image of $W(F)$ [L2, Theorem 7.2]. In [C2] it is shown that any formally real field $F$ is contained in an order closed field $K$ satisfying SAP with restriction of orderings giving a homeomorphism $X_K \rightarrow X_F$. However, it is not always possible to find such a $K$ which is algebraic over $F$. Following [P, Theorem 9.1], we use the following valuation theoretic characterization of SAP fields:

(5.2). For each place $\sigma: K \to \mathbb{R} \cup \{\infty\}$ with value group $\Gamma$, $|\Gamma/2\Gamma| \leq 2$, and if $|\Gamma/2\Gamma| = 2$, then the residue field has a unique ordering.

**Example 5.3.** Let $F = \mathbb{Q}((x))((y))$, the field of iterated Laurent series in two variables over the rational numbers. The field $F$ has four orderings and the ring $W_{\text{red}}(F)$ is generated by $1$ and the functions $2^\chi_U$, where $\chi_U$ ranges over the characteristic functions of all subsets consisting of two orderings. In particular, no single ordering can be separated from the other three and the field is not SAP. We consider two extension fields of $F$. First consider $F_1 = F(\sqrt{2}, \sqrt{3}, (x\sqrt{2})^{1/2^n}, (y\sqrt{3})^{1/2^n}, n = 1, 2, \ldots)$. It is easily seen that the orderings of $F$ extend uniquely to $F_1$ since $x$ and $\sqrt{2}$ (respectively, $y$ and $\sqrt{3}$) are forced to always have the same sign. Thus an order closure $K_1$ of $F_1$ is also an order closure of $F$. In [C2, Example 3] the field $K_1$ is used to illustrate the construction of a SAP order closed field. On the other hand, the field $F_2 = R((x))((y))$, where $R$ consists of all real algebraic numbers, is also an algebraic extension of $F$ to which each ordering extends uniquely. But $F_2$ is henselian, so its unique place to $\mathbb{R}$ extends uniquely to a real place on any order closure $K_2$. The residue field $R$ is real closed and algebraic over $\mathbb{Q}$, hence has a unique homomorphism into the real numbers. Thus $K_2$ has only one place into $\mathbb{R}$ and hence has value group $\Gamma$ satisfying $|\Gamma/2\Gamma| = 4$. It follows from (5.2) that $K_2$ is not SAP, and therefore $W_{\text{red}}(F) \to W(K_2)$ is an isomorphism since there are only two possible reduced Witt rings (up to isomorphism) for a field with four orderings [C5].

We next take a more careful look at the special case where $F$ is a rational function field. We begin with some preliminaries.

**Lemma 5.4.** Let $F \subset K$ be formally real fields.
(a) The induced homomorphism $W_{\text{red}}(F) \to W_{\text{red}}(K)$ is injective iff $X_K \to X_F$ is surjective.

(b) If the induced homomorphism $W_{\text{red}}(F) \to W_{\text{red}}(K)$ is surjective, then $X_K \to X_F$ is injective, but not conversely.

Proof. Two of the three implications follow immediately from considering the fact that the reduced Witt ring may be considered as a subring of continuous functions from the space of orderings to the ring of integers. The remaining implication is the forward direction of (a) where we assume that $W_{\text{red}}(F) \to W_{\text{red}}(K)$ is injective. Given any ordering of $F$, it corresponds to a minimal prime ideal $p$ of $W_{\text{red}}(F)$. Since $W_{\text{red}}(K)$ is integral over $W_{\text{red}}(F)$, the going-up theorem implies that there exists a prime ideal $p'$ of $W_{\text{red}}(K)$ lying over $p$. It is minimal since $p' \cap Z \subseteq p \cap Z = \{0\}$, and hence corresponds to an ordering of $K$ mapping onto the given ordering of $F$. The fact that the converse of (b) fails was illustrated by the embedding of $F$ in $K_1$ of Example 5.3.

To analyze rational function fields, we must know how their real valuations correspond to orderings. The following facts come from [P, Chap. 9; C3]. Let $F$ be an ordered field with real closure $R$. Corresponding to each element $r$ in $R$ there are a pair of orderings of $F(x)$ extending the given ordering on $F$. In these orderings $x$ is infinitesimally close to $r$, greater than $r$ in one and less than $r$ in the other. If $f(x)$ is the minimal polynomial of $r$ over $F$, then the valuation ring for a compatible real valuation is $F[x]$ localized at the prime ideal generated by $f(x)$. We shall denote this valuation by $v_x$. Note that it is trivial on $F$ and the residue field is $F(r)$. If $f$ has another root $s$ in $R$, then the valuation $v_s$ is the same as $v_r$.

**Theorem 5.5.** Let $F$ be a formally real field and let $K$ be any formally real extension field of the rational function field $F(x)$. Assume that $W_{\text{red}}(F(x)) \to W_{\text{red}}(K)$ is surjective.

(a) Let $f \in F[x]$ be irreducible with a root $\alpha$ in $K$. Assume that $f$ has a root $\beta \neq \alpha$ in some real closure $R$ of $F$ with respect to an ordering which extends to $K$. Then if the valuations $v_\alpha$ and $v_\beta$ extend to real valuations $w_\alpha$ and $w_\beta$ on $K$, at least one of the value groups $\Gamma_\alpha = w_\alpha(K)$ or $\Gamma_\beta = w_\beta(K)$ must be 2-divisible.

(b) If $W_{\text{red}}(F(x)) \to W_{\text{red}}(K)$ is also injective, then every irreducible polynomial over $F$ with a root in $K$ has exactly one root in every real closure of $F$.

(c) If $K$ is a finite extension of $F(x)$, then $W_{\text{red}}(F(x)) \to W_{\text{red}}(K)$ is injective and the extension has odd degree.

Proof. (a) Since $\alpha \in K$, the elements $\alpha$ and $\beta$ have distinct minimal
polynomials over $K$ and hence $w, \neq w_\beta$. If neither $\Gamma_x$ nor $\Gamma_\beta$ is 2-divisible, then each valuation $w, x$ and $w_\beta$ has at least two compatible orderings, say $P^+, P^-$ corresponding to $w, x$ and $Q^+, Q^-$ corresponding to $w_\beta$. We may choose all four so that they induce the same ordering on $F$ as $R$ does. By Lemma 5.4, the restriction mapping $X_F \rightarrow X_{F(x)}$ is injective, so these four orderings restrict to four orderings of $F(x)$ which are separated in pairs by the polynomial $f$. In $X_{F(x)}$ no one can be separated from the other three since $f$ is irreducible. But in $X_F$ the polynomial $x - x$ lies in either one or three of the orderings (one if $x > \beta$, three if $x < \beta$). This contradicts the surjectivity of the Witt ring mapping.

(b) Assume an irreducible polynomial $f$ in $F[x]$ has two roots $x, \beta$ in some real closure $R$ of $F$ with at least one in $K$. By (a), one of the value groups $\Gamma_x$ or $\Gamma_\beta$ is 2-divisible, say $\Gamma_x$. Fix the ordering on $F$ induced by $R$. Extending this, the field $F(x)$ has two orderings corresponding to the place carrying $x$ to $x$, but $\Gamma_x$ being 2-divisible implies that only one extends to $K$. However, our hypothesis of an isomorphism between the reduced Witt rings implies by Lemma 5.4 that every ordering of $F(x)$ extends to $K$, a contradiction.

(c) Since the extension is finite, it is generated over $F(x)$ by a single element $x$ in $K$. Let $f$ be the minimal polynomial of $x$ over $F(x)$. Fix any ordering of $F(x)$ which extends to $K$ and let $R$ be the real closure of $F(x)$ with respect to that ordering. Applying Lemma 5.4, we see that the ordering extends uniquely to $K$. But the number of such extensions equals the number of embeddings of $K$ into $R$. Thus $x$ is the only root of $f$ in $R$, and therefore $f$ has odd degree. It follows that every ordering of $\Gamma(x)$ extends to $K$ since the extension is of odd degree, and therefore $X_F \rightarrow X_{F(x)}$ is a homeomorphism. Applying Lemma 5.4 once more gives the desired conclusion.

**Theorem 5.6.** Let $F$ be a formally real field and let $L$ be a strong order closure of $F$. The following conditions are equivalent:

(a) $W_{\text{red}}(F(x)) \rightarrow W_{\text{red}}(L(x))$ is an isomorphism.

(b) Every irreducible polynomial over $F$ with a root in $L$ has a unique root in each real closure of $F$.

(c) Every irreducible polynomial of odd degree over $F$ has a unique root in each real closure and $[L:F]$ is odd (as a supernatural number $[R]$).

(d) $[L:F]$ is odd (as a supernatural number).

(e) For any intermediate field $K$ between $F$ and $L$, each ordering of $F$ extends uniquely to $K$ and $K$ is pythagorean.

(f) For any intermediate field $K$ between $F$ and $L$, each ordering of $F$ extends uniquely to $K$.
Proof. (a) ⇒ (b). This follows immediately from Theorem 5.5(b).

(b) ⇒ (c). Since \( L \) is an order closure of \( F \), every irreducible polynomial of odd degree necessarily has a root in \( L \) and thus has a unique root in every real closure. If \( L \) contains any element of even degree over \( F \), its minimal polynomial over \( F \) will have at least two roots in each real closure, contradicting (b).

(c) ⇒ (a). As shown in [C3], the reduced Witt ring of a rational function field is generated by 1 and \( 2_{\mathcal{U}} \), where \( U \) ranges over sets \( X(f) \) with \( f \) an irreducible polynomial. Furthermore, if \( f \) is the minimal polynomial of an element \( \alpha \) in \( L \), then [C3, Sect. 4] implies \( X(f) = X(x - \alpha) \) as subsets of \( X_{F(\alpha)} \) and \( X_{L(\alpha)} \), which we identify under the canonical homeomorphism, because \( \alpha \) is the only root of \( f \) in each real closure. It follows that the reduced Witt rings of the rational function fields are isomorphic.

(c) ⇒ (d). Trivial.

(d) ⇒ (c). Since \( K \) is between \( F \) and an order closure of \( F \), every ordering of \( F \) must extend to \( K \). If \( K \) has two orderings which restrict to the same ordering of \( F \), one must be killed in extending to \( L \). But this contradicts the fact that \( L \) is an odd extension of \( K \). Thus every ordering of \( F \) extends uniquely to \( K \). Since \( L \) is pythagorean, any sum of squares \( \alpha \) in \( K \) is a square in \( L \). But then \( K(\sqrt{\alpha}) \subset L \). Since \([L:K] \) is odd, we must have \( \sqrt{\alpha} \in K \).

(e) ⇒ (f). Trivial.

(f) ⇒ (b). Let \( f \) be an irreducible polynomial over \( F \) with root \( \alpha \) in \( L \). Given any real closure \( R \) of \( F \), the induced ordering on \( F \) extends uniquely to \( F(\alpha) \). This implies that \( F(\alpha) \) has a unique embedding into \( R \) and hence \( f \) has a unique root in \( R \).

Remark 5.7. (a) Recall that a field is called hereditarily euclidean if every formally real extension of it is euclidean and is called (absolutely) hereditarily pythagorean if every formally real extension is pythagorean [Be]. These give examples of fields satisfying the theorem above. We verify condition (f) as follows: if \( F \) is hereditarily pythagorean, then any maximal odd degree extension \( L \) is clearly SOC. Any subfield of \( L \) containing \( F \) has odd degree over \( F \), and hence every ordering of \( F \) extends uniquely by [Be, Corollary 1, p. 90].

In general, a strongly order closed field is not hereditarily pythagorean. In fact, any hereditarily pythagorean field has a henselian valuation with respect to which the residue field is again hereditarily pythagorean and has at most two orderings. Thus if \( F \) is a SOC field with more than two orderings, it will generally not be hereditarily pythagorean. On the other hand, if it has at most two orderings, it will be hereditarily pythagorean.
This is clear if it has only one ordering, since it is real closed. Assume the SOC field $F$ has two orderings. Since any minimal extension of $F$ is quadratic, it will suffice to show that any formally real quadratic extension of $F$ is again pythagorean, and hence SOC. But a pythagorean field with two orderings is superpythagorean [Br1, Theorem 1] and hence any quadratic extension is pythagorean [Be, Theorem 2, p. 89].

(b) Theorem 5.6 should be considered as a generalization of the well-known case when $F$ has a unique ordering. In this case the strong order closure is real closed, so $L(x)$ satisfies SAP. The isomorphism of reduced Witt rings says that $F(x)$ satisfies SAP and therefore $F$ is hereditarily euclidean [C3, Theorem 15; P, Theorem 9.8]. The remaining conditions of Theorem 5.6 specialize to known characterizations of hereditarily euclidean fields. In the next corollary we carry this case a little further.

**Corollary 5.8.** Assume that $F$ has a unique ordering and let $K$ be a strong order closure of $F(x)$. Then the following conditions are equivalent:

(a) $W_{\text{red}}(F(x)) \rightarrow W(K)$ is an isomorphism.

(b) $F(x)$ satisfies SAP.

(c) $F$ is hereditarily euclidean.

**Proof.** By Theorem 5.6 and the remark above, we have the equivalence of (b) and (c) with the isomorphism $W_{\text{red}}(F(x)) \rightarrow W_{\text{red}}(R(x))$, where $R$ is a real closure of $F$. When these fields are SAP, the reduced Witt rings are maximal for the given space of orderings, and hence they are also isomorphic to $W(K)$. Conversely, a strong order closure of $F(x)$ must contain a strong order closure of $F$; that is, a real closure $R$. It follows that $W(K)$ must be SAP and hence, from the isomorphism of reduced Witt rings, that $F(x)$ is SAP.

**References**


