THE RUNNER'S PARADOX: A MEAN-VALUE THEOREM FOR INTERVALS

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The following query was posted to an internet running group: "I recently ran a local 12 kilometer race and finished in exactly 48 minutes. Can I now brag that I have run 10 kilometers in 40 minutes (or less)?" Unfortunately for our would-be boastful running friend, it turns out that this is not the case. Indeed, as we will illustrate in Example 2, it is not hard to create continuous functions (representing distance the runner traveled as a function of time) for which every 10-kilometer segment of the race would have been run in more than 40 minutes (and likewise, functions for which every 10-kilometer segment would have been run in less than 40 minutes). Without more information on how the runner paced his race, it is impossible for him to make this claim.

In general, the question being asked is the following: Let f be a continuous function defined on some interval of length l, and let s be a fixed length less than l. Does there exist a subinterval of length s on which the average rate of change of f equals the average rate of change of f on the entire interval? Thus it is a kind of "mean-value theorem" for intervals.

Not surprisingly, for sufficiently well-behaved functions such an interval will always exist. For instance, if our running friend had started out slowly and had steadily picked up the pace, or had started out fast and had steadily slowed down, thus staying either always behind, or always ahead, of a runner sticking to a steady 4 minutes per kilometer pace, then he would have run an n-kilometer subinterval of the race in $4n$ minutes for every positive real number $n \leq 12$, as the following theorem shows.

Theorem 1. *Let f be a continuous function on $[a, b]$. Assume that the graph of f does not cross the line through $(a, f(a))$ and $(b, f(b))$. Let $0 < s < b - a$ be fixed. Then there exists a subinterval $[c, d]$ of $[a, b]$ of length s for which $\frac{f(d)-f(c)}{d-c} = \frac{f(b)-f(a)}{b-a}$; i.e. such that the average rate of change on the subinterval equals the average rate of change on the entire interval.*

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Proof. As in the usual calculus proof of the mean-value theorem, we may assume that $f(a) = f(b) = 0$. We may assume that $f(x) \geq 0$ on (a, b), since otherwise one can replace $f(x)$ by $f(-x)$. We consider the continuous function $h(x) = f(x+s) - f(x)$ defined on $[a, b-s]$. If $h(x)$ is ever zero, we are done. But $h(a) = f(a+s) \geq 0$ and $h(b-s) = -f(b-s) \leq 0$; if neither of these is zero, then the intermediate-value theorem gives a point in $(a, b-s)$ at which h is zero. □

Suppose, however, that the runner is a bit more like the proverbial hare, first bursting ahead of the steady 4-minutes-per-kilometer runner, then lagging behind him. Under these circumstances it may well happen, depending on s, that for each interval of length s, he will be running more slowly (or running faster) on that interval than the average pace for the entire distance, as the following example illustrates.

Example 2. **(a)** Consider the function $f(x) = \sin(2\pi x)$ on $[0, 1]$. One sees that for any $s > 1/2$, the average rate of change of f on an interval of length s is negative, while the average rate of change on $[0, 1]$ is 0.

(b) Similarly, consider the function $f(x) = -\sin(2\pi x)$ on $[0, 1]$. One sees that for any $s > 1/2$, the average rate of change of f on an interval of length s is positive, while the average rate of change on $[0, 1]$ is 0.

The situation of Example 2 can happen for most values of s. It fails to be possible only when the length of the entire interval is a positive integer multiple of the length s of the subinterval. Thus the runner could indeed claim that he had run some six-kilometer stretch of his race in exactly 24 minutes, some three-kilometer stretch in exactly 12 minutes, and some one-kilometer segment in exactly 4 minutes. This is shown by the following theorem, which can be found in [1, page 98], and is a consequence of the universal chord theorem proved by P. Levy [4].

Theorem 3. *Fix $0 < s < 1$. One can find a continuous function f on $[0, 1]$ such that the average rate of change of f on every interval of length s is always greater than the average rate of change of f on $[0, 1]$ if and only if s is not the reciprocal of an integer.*

Proof. Once again we may assume that $f(0) = f(1)$, so that the average rate of change of f on $[0, 1]$ is 0. If s is the reciprocal of an integer, the claim is easily seen as the average of f on each subinterval $[ks, (k+1)s]$ being positive implies that the average on the entire interval is positive, not zero. On the other hand, if s is not the reciprocal of an integer, set $B = |\sin(\pi/s)| > 0$. Define $f(x) = Bx - |\sin(\pi x/s)|$ on $[0, 1]$. Then $f(0) = 0$ and $f(1) = B - |\sin(\pi/s)| = 0$ and $f(x+s) - f(x) = Bs$, so the average rate of change on any interval of length s is $B > 0$. □

Example 4. Even when s is the reciprocal of an integer, it may happen that the average rate of change on each subinterval of length s is always greater than or equal to the average rate of change on the entire interval and is sometimes strictly greater.

For example, for $s = 1/3$ and $k > 0$, one can consider the function

$$f_k(x) = \begin{cases} k\sin(6\pi x), & \frac{1}{6} \le x \le \frac{5}{6} \\ 0, & \text{otherwise.} \end{cases}$$

The average rate of change of f_k on the subintervals of length $1/3$ ranges from 0 to $3k$, although the average rate of change on the entire interval is 0.

We need to know more about the values of s that work when f is an unrestricted continuous function. It was shown in [1] (see also [2]) that at least half the potential values of s work. In fact, the proof there shows that for any $0 < s < b - a$, there will exist an interval of length either s or $(b - a) - s$ (possibly both) on which f will have the desired average rate of change. Example 2 shows that in general one cannot hope for more than half the values of s to work.

On the other hand, for any given continuous function, there is some interval $(0, \varepsilon]$ such that for any s in this interval, one can always find a subinterval of length s with the desired average rate of change. Indeed, working with the normalized case (interval $[0, 1]$, $f(0) = f(1) = 0$), just let c be a place at which $f(c) \neq 0$ (assuming f is not identically zero). Then there exists an interval $[c - \varepsilon/2, c + \varepsilon/2]$ on which f is nonzero. For every $s \in (0, \varepsilon]$, it is clear that we can find $x, y \in [c - \varepsilon/2, c + \varepsilon/2]$ such that $f(x) - f(y) = 0$ and $y - x = s$.

To have some control over the size of ε, we must know more about the function f. For example, if f has exactly n zeros in $(0, 1)$, then one of the intervals between zeros must have length at least $1/(n + 1)$, so Theorem 1 says that $\varepsilon = 1/(n + 1)$ will work. In fact, we can do somewhat better. We give the precise bounds found by Levit [3] in the following theorem.

Theorem 5. *Assume the continuous function f changes sign exactly $n \geq 0$ times on the interval $(0, 1)$, $f(0) = f(1) = 0$, and f is not identically zero on any subinterval.*

(1) *If n is odd, then for any $s \in (0, \frac{2}{n+3}]$, there exists a subinterval of $[0, 1]$ of length s on which the average rate of change of f is zero.*
(2) *If n is even, then for any $s \in (0, \frac{2}{n+2}]$, there exists a subinterval of $[0, 1]$ of length s on which the average rate of change of f is zero.*
(3) *The intervals in (1) and (2) cannot be enlarged.*

References

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