ORDERINGS, VALUATIONS AND HERMITIAN FORMS OVER $\ast$ -FIELDS

THOMAS C. CRAVEN

1. Introduction and notation. Let D be a division ring (also called a skew field) with an involution $\ast$ (an anti-automorphism of order 2). We shall refer to $(D, \ast)$ as a $\ast$ -field. The purpose of this paper is to provide a guide to the literature concerning orderings in $\ast$ -fields and their relationships with valuations and hermitian forms. Most of the topics are generalizations of results in the theory of formally real fields, which have been very nicely presented by T. Y. Lam in [L]. Indeed, if the involution $\ast$ is the identity, then D is necessarily commutative, so we have a direct generalization. This paper should be of particular benefit to those who are already familiar with the commutative theory and wish to see how it might be generalized. Applications so far are primarily to the theory of division rings, but there are potential future applications to $C^\ast$ -algebras (see [Ha]).

Examples of $\ast$ -fields. When $\ast$ is not the identity, commutative examples have the form of a field extension K/F of degree 2, where the involution is the nontrivial automorphism generating the Galois group. The simplest noncommutative examples are quaternion algebras, generated, say, by i, j over a field F , with involution given by $i^\ast = -i, j^\ast = -j$; we refer to such algebras as *standard quaternion algebras*. These examples play a rather trivial role in the theory as most results follow readily from a knowledge of the theory of formally real fields.

The interesting examples which have been constructed consist of nonstandard quaternion algebras (a nonstandard involution causes much different behavior), tensor products of such algebras, crossed products, skew fields of polynomials or Laurent series, and some particularly important special constructions [CW]. Further specific sources of examples (with discussion of orderings and valuation theory) can be found in [D], [G] and [MW] as well as in most of the referenced papers of Chacron, Craven and Holland.

Notation. We write $S(D) = \{d \in D \mid d^\ast = d\}$ for the set of symmetric (self-adjoint) elements of $(D, \ast)$ and denote the center of D by Z_D . For any subset $A \subseteq D$, we shall use the notation $A^\times$ to denote the set of nonzero elements in A . We write multiplicative

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commutators in the form $[a, b] = aba^{-1}b^{-1}$, and, more generally, for subsets $A, B \subseteq D^\times$, we write $[A, B] = \{[a, b] \mid a \in A, b \in B\}$. As we shall see in the next section, a basic idea behind orderings is that all norms (elements of the form dd^* for $d \in D^\times$) should be positive. These replace squares in the formally real case. In some situations, certain commutators must also be positive, so we define $\Sigma(D)$ to be the set of all sums of products of norms and elements of $[D^\times, S(D)^\times]$.

Structure theory for $$ -fields.* As we shall see in the next section, some types of orderings only exist in skew fields which are purely transcendental over their centers, while others exist in division rings finite-dimensional over their centers. There is very little known structure theory in the former case. We mention two general facts. One is that either $Z_D \subseteq S(D)$ (in which case, the involution is said to be of the *first kind*) or the center is a quadratic extension of the field of central symmetric elements (in which case, the involution is said to be of the *second kind*). The second is a particularly useful result due to Dieudonné [H1, Lemma 1]: if $(D, *)$ is noncommutative and not a standard quaternion algebra, then the symmetric elements generate D as a ring.

The existence of certain types of orderings puts strong conditions on finite-dimensional $*$ -fields. Some of these results can be found in [CW] (see §3 below) and [I2] (see §2 below).

2. Orderings. In this section we shall describe the many different definitions of ordering and semiordering which have been proposed for a $*$ -field. No definition can give a total ordering of D since skew elements (i.e., $d^* = -d$) can be neither positive nor negative. All definitions give a total ordering of $S(D)$. The definitions vary in whether they are restricted to $S(D)$ or are defined on larger subsets of D . More importantly, they vary in the form of their multiplicative properties. Recall that in a formally real field, a semiordering differs from an ordering in requiring closure only under multiplication by squares [Pr, p. 5].

We shall make an effort to standardize terminology and notation so that the relationships among the various definitions are more readily apparent. All orderings will be defined in terms of their positive cone, a subset $P \subset D$, with the understanding that $a < b$ in D is equivalent to $b - a \in P$. Most definitions come in pairs, one for an ordering with $P \subset S(D)$, and the other an “extended” version where some nonsymmetric elements are ordered as well. The relationship between these versions is not always apparent. The most important role of the extended versions is usually to aid in proofs. All definitions are designed so that the set $P = \mathbb{R}^+$ of positive real numbers works for each of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (real quaternions).

The subject began with a definition given by Baer [B, Chap. IV, App. I]. This definition has the advantages that it works well with hermitian forms (see §4) and applies to finite-dimensional division algebras. Our first few definitions all define a semiordering in the case where D is commutative and $*$ is the identity.

Definition 2.1. A *Baer ordering* on $(D, *)$ is a subset $P \subseteq S(D)^\times$ satisfying $P + P \subseteq P$, $dPd^* \subseteq P$ for all $d \in D^\times$, $1 \in P$, and $P \cup -P = S(D)^\times$.

As noted in [Ch1], the same definition can be used without restricting P to be a subset of $S(D)$. When needed, we shall call this an *extended Baer ordering*. An alternative to this definition of ordering was suggested by Chacron [Ch1], the difference being in how the norm dd^* is used in the multiplicative requirement. It was initially proposed in the following extended version.

Definition 2.2. A *c-ordering* on $(D, *)$ is a $*$ -closed subset $P \subseteq D^\times$ satisfying $P + P \subseteq P$, $Pdd^* \subseteq P$ for all $d \in D^\times$, $1 \in P$, and $P \cup -P \supseteq S(D)^\times$.

The following strengthening of the axioms for a c-ordering was proposed in [Ch3, p.17] to make the valuation theory work better (see §3).

Definition 2.3. A *normal c-ordering* is a c-ordering P with the property that, for any element $u \in S(D)$ which is bounded above and below by positive rational numbers, one has $sus \in P$ for all $s \in S(D)^\times$.

More recently, Chacron [Ch4] introduced the restriction of a c-ordering to the symmetric elements. A formal definition is given by

Definition 2.4. A *symmetric c-ordering* on $(D, *)$ is a subset $P \subseteq S(D)^\times$ satisfying $P + P \subseteq P$, $a(dd^*) + (dd^*)a \in P$ for all $a \in P$, $d \in D^\times$, $1 \in P$, and $P \cup -P = S(D)^\times$.

Holland [H4] was the first to notice that it is quite reasonable to require closure under multiplication also, thus introducing the first true orderings (which he called *strong orderings*). These satisfy the axioms of both Baer orderings and c-orderings. One disadvantage of these orderings is that, just as with total orderings of a skew field without involution, such orderings can exist in a noncommutative skew field D only if D is purely transcendental over its center or D is a standard quaternion algebra.

Definition 2.5. An *extended *-ordering* on $(D, *)$ is a $*$ -closed subset $P \subseteq D^\times$ satisfying $P + P \subseteq P$, $dPd^* \subseteq P$ for all $d \in D^\times$, $1 \in P$, $P \cup -P \supseteq S(D)^\times$, and $P \cdot P \subseteq P$.

Note that an extended *-ordering P is a normal subgroup of $D^\times$; indeed, if $x, y \in P$, then so are $x^{-1} = x^*(x^{-1}*x^{-1})$ and $xyx^{-1} = (xyx^*)(yx^*)^{-1}$. The restricted version of this was introduced by Idris [I1], who called it a *Jordan ordering*. It is simply a Baer ordering closed under a symmetric version of multiplication.

Definition 2.6. A **-ordering* on $(D, *)$ is a Baer ordering P satisfying $xy + yx \in P$ for all $x, y \in P$.

The next two definitions were introduced by Craven [Cr4]; they give a version of semiordering which works well with *-orderings, in that virtually all results concerning

semiorderings in the commutative theory can be extended to these. Again, their existence implies that D is infinite-dimensional over its center. As in the commutative theory, it is also possible to make the definitions relative to a preordering (cf. [Cr4, §3]) in place of $\Sigma(D)$.

Definition 2.7. A **-semiordering* on $(D, *)$ is Baer ordering P satisfying $xy + y^*x \in P$ for all $x \in P$ and $y \in \Sigma(D)$.

Using $\prod S(D)$ to denote the set of all nonzero products of symmetric elements, we can state

Definition 2.8. An *extended *-semiordering* on $(D, *)$ is an extended Baer ordering P satisfying $P \cup -P \supseteq \prod S(D)$ and $P \cdot \Sigma(D) \subseteq P$.

The final definition to be considered here was introduced by Leung [Le1]. It has stronger multiplicative properties than a Baer ordering, but still can exist in the case of division rings finite-dimensional over their centers. For this finite-dimensional situation, the orderings are determined by their restriction to the center, so the commutative theory extends more easily. In the infinite-dimensional case, little is known.

Definition 2.9. A *weak *-ordering* on $(D, *)$ is a Baer ordering P satisfying $aP = P$ for any $a \in P \cap Z_D$.

Archimedean orderings. The earliest work with these orderings gives characterizations of the archimedean ones. Holland [H1, Theorem 2] shows that every archimedean Baer ordered $*$ -field is $*$ and order isomorphic to a subfield of the real numbers, complex numbers or real quaternions. The same holds true for a c-ordered field since archimedean c-orderings are Baer orderings (cf. [Ch1]).

Artin-Schreier theory. The most fundamental question to ask regards the extension of Artin-Schreier theory on existence of orderings and intersections of orderings. Note that sums of nonzero norms are positive in all types of orderings. However, to obtain the existence of an ordering (any type), it is not sufficient to require that zero cannot be written as a sum of nonzero norms [H1, Corollary 5]. An alternative is to require that $\sum d_i s d_i^* \neq 0$ whenever $s \in S(D)^\times$ and $d_i \in D^\times$. It remains an open question as to whether this condition is sufficient to force D to even have a Baer ordering. Leung [Le2, Theorem 2.9] is able to get results which formally resemble Artin-Schreier theory, but the defined objects are not sufficiently computable to resolve this question.

For c-orderings, the situation is somewhat better. If zero cannot be written as a sum of products of nonzero norms, then D has a c-ordering [Ch1, Theorem 2] by the usual Zorn's lemma argument. Unfortunately, no equivalent condition is known for the existence of symmetric c-orderings.

For *-orderings and extended *-orderings, the usual Artin-Schreier results still hold (cf. [Ha] in the commutative case and [H4] for extended *-orderings). The intersection of all extended *-orderings is $\Sigma(D)$ and the intersection of all *-orderings is $\Sigma(D) \cap S(D)$. Idris [I1] has shown (see §3 for details) that every *-ordering extends uniquely to a maximal extended *-ordering, so it is easy to correlate results between the two types of orderings.

Extension questions. For both *-orderings and c-orderings, very little is known about the collection of extended ones which intersect with $S(D)$ to give a particular restricted one. The following example shows that there may be infinitely many such, even when D is commutative (where extended *-orderings are, in fact, c-orderings).

Example 2.10. Let $K = \mathbb{R}((x))$ be the field of formal Laurent series over the real numbers and let F be the subfield $\mathbb{R}((x^2))$. Let $*$ be the nontrivial automorphism of K over F (i.e. $f(x)^* = f(-x)$). Note that $S(K) = F$. As is well known, the field F has exactly two (ordinary) orderings. Let $P \subseteq F$ be the ordering of F in which x^2 is negative; that is, $P = \{ \sum_{i=-n}^{\infty} a_i x^{2i} \mid (-1)^n a_{-n} > 0 \}$. Since, for $f(x) = \sum_{i=-n}^{\infty} a_i x^i \in K$, we have $f(x)f(x)^* = f(x)f(-x) = (-1)^n a_{-n}^2 x^{-2n} + \dots \in P$, the set P is a *-ordering of K . The largest extended *-ordering containing P is the set $P_{max} = \{ \sum_{i=-n}^{\infty} a_i x^i \in K \mid n \text{ is even and } (-1)^{n/2} a_{-n} > 0 \}$, obtained from a study of the associated valuation theory (see Theorem 3.2). It is not difficult to check that there are infinitely many extended *-orderings between P and P_{max} , namely, all sets P_t , where t is an odd integer and P_t consists of all finite sums of elements from $\bigcup_{n=0}^{\infty} (1 \pm x^t)^n P$.

Topology. As for orderings of commutative fields, one can topologize the sets of Baer orderings, *-orderings, c-orderings, etc. The actual work which has been done in this regard is only for Baer orderings [Cr2], [Le2], *-orderings [Cr2], [Cr4], [Cr5], *-semiorderings [Cr2] and weak *-orderings [Le2]. In all of these cases, the topology is generated by the subbasis consisting of all sets $H(a)$, defined to be the set of all orderings in which $a \in S(D)^\times$ is positive. As in the commutative case, the spaces are compact, Hausdorff and totally disconnected. In the case of *-orderings, this topology can also be obtained as the Zariski topology on $WR(D, *)$, a ring of equivalence classes of hermitian forms defined in [Cr4] (see §4 for more information). In the sequel, we shall write X_D for the topological space of all *-orderings on $(D, *)$.

One of the earliest questions considered (in the commutative theory) regarding this topology was to ask when every clopen (both closed and open) subset can be realized as a set of the form $H(a)$ for some $a \in S(D)$ (cf. [KRW, Theorem 3.20], [Pr, §9], [L, §17]). This is known as the *strong approximation property (SAP)*. Most of the many equivalent conditions for commutative fields can be generalized. For example, the space of *-orderings X_D satisfies SAP if and only if every *-semiordering is actually a *-ordering [Cr2, Theorem 4.1]. And, for the case of D being finite-dimensional over its center, the space of Baer orderings satisfies SAP if and only if every Baer ordering is actually a weak *-ordering; furthermore, in this case, D either is commutative, a standard quaternion algebra, or $[D : Z_D]$ is odd (see [Le2, §5]).

We introduce “fans” here primarily because they will be needed in §4. However, they are also related to the strong approximation property. We shall make the definitions in the $*$ -ordering situation. A *preordering* is a subset $T \subseteq S(D)^\times$ satisfying $T+T \subseteq T$, $dTd^* \subseteq T$ for all $d \in D^\times$, $S(D) \cap \Sigma(D) \subseteq T$, $1 \in T$ and $xy + y^*x^* \in T$ whenever $x + x^*$, $y + y^* \in T$ for x, y in the subgroup $S(D)^\times \cdot \Sigma(D)$ of $D^\times$ (see [Cr4, §3]). A preordering is called a *fan* if, for all $a \in S(D)^\times$, if $a \notin -T$, then $1 + a \in T \cup aT$ (see [Cr3, Theorem 2.8]). For any preordering T , we write $X_T = \{P \in X_D \mid P \supseteq T\}$. Then SAP holds if and only if $|X_T| \leq 2$ for all fans T . Indeed, if $|X_T| \geq 4$, then there exist four $*$ -orderings in X_T such that three of them cannot be separated from the fourth by any element $a \in S(D)$. If such separation is always possible, then one can show that SAP holds (see Theorem 4.3). A similar result holds for Baer orderings when D is finite-dimensional [Le2, Theorem 5.6].

3. Valuations and orderings. For commutative fields, every ordering or semiordering has an associated valuation ring; in fact, one for which the residue class field has an induced archimedean ordering or semiordering. Noncommutativity creates problems which are, in some respects, partially compensated for by the involution.

Let P be a Baer ordering of $(D, *)$. To P we associate the set $A(P) = \{d \in D \mid n - dd^* \in P \text{ for some } n \in \mathbb{Z}\}$. This was first studied in [H2], where it is shown that $A(P)$ is a total subring of D (i.e., $d \in A(P)$ or $d^{-1} \in A(P)$ for every nonzero $d \in D$), with unique (two-sided) maximal ideal of infinitesimal elements $\mathfrak{m}_P = \{d \in A(P) \mid q - dd^* \in P \text{ for all } q \in \mathbb{Q}^+\}$. To be a valuation ring in the sense of Schilling [Sg], the ring $A(P)$ must also be invariant under inner automorphisms of D . All the early examples of Baer orderings were, in fact, invariant leaving the existence of noninvariant ones open for several years. Morandi and Wadsworth [MW] have constructed examples in every possible finite dimension (of D over its center) of Baer orderings for which $A(P)$ is not invariant. In any case, the residue (skew) field has an induced involution and an induced archimedean Baer ordering, making it isomorphic to a sub- $*$ -field of the real quaternions $\mathbb{H}$. Holland [H2, §4] showed that there is an associated *order valuation* v , but that the set of values Γ_v is not generally a group unless v is a $*$ -valuation ($v(d) = v(d^*)$ for all $d \in D$). In this case it is necessarily an abelian group since $v(a) + v(b) = v(ab) = v(b^*a^*) = v(b) + v(a)$.

Compatibility. We shall follow [L] and [Cr4, §4] in defining compatibility here. We say a $*$ -valuation v and a Baer ordering P are *compatible* if the valuation ring A_v is convex with respect to P , and are *fully compatible* if $0 < a \leq b$ with respect to P implies $v(a) \geq v(b)$ in Γ_v . The latter, stronger definition is the one often used for compatibility in the literature. It is worth making a distinction since any Baer ordering is compatible with its order valuation, but may not be fully compatible with any valuation, even in the commutative case with identity involution [Pr, Theorem 7.17]. For $*$ -orderings, the conditions are equivalent [Cr4, Theorem 4.2].

Lifting theorems. For a $*$ -valuation v , lifting Baer orderings from the residue field back to D encounters the following complication [H2], [Cr1]. For any nonzero symmetric element $s \in S(D)$, there is an involution $\#$ induced on the residue field D_v defined by $y^\# = \overline{xs}^{-1}$,

where $x \in D$, $\bar{x} = y \in D_v$. The element s is called *smooth* if the induced involution $\#$ is conjugate to $*$ (induced on D_v by the involution $*$ on D). The valuation is called *smooth* if each equivalence class $v^{-1}(\gamma)$, $\gamma \in \Gamma_v$, which contains some symmetric element, contains a smooth symmetric element. For a smooth valuation v , one can completely characterize the set of Baer orderings of D fully compatible with v by generalizing the usual theorems lifting orderings from the residue field. See [Pr, Theorem 7.8] for the case of semiorderings of commutative fields with identity involution and [C1, Theorem 3.4] for the general case. A simpler situation occurs when D is of odd dimension over its center (see [Le2, Prop. 3.5, 3.6]).

With a slightly stronger version of smoothness to handle products (cf. [C1, Definition 2.1]), this characterization can also be carried out for $*$ -orderings [C1, Theorem 4.3]. To obtain a theorem which is easier to state, we consider a stronger condition on the valuation; we say a $*$ -valuation is *real* if the commutator subgroup $[D^\times, S(D)^\times]$ is mapped into some extended $*$ -ordering of the residue field. This is equivalent to requiring that all $*$ -orderings lift [Cr2, Proposition 2.2], or that $[D^\times, S(D)^\times]$ maps into $\Sigma(D_v)$ [Cr3, Proposition 2.1]. In particular, the order valuation of a $*$ -ordering is always real [H4, §5].

Theorem 3.1 [Cr3, Theorem 2.2]. *Let v be a real $*$ -valuation on $(D, *)$. Let $S(\Gamma_v) = \{\gamma \in \Gamma_v \mid \gamma = v(s) \text{ for some } s \in S(D)\}$. Then there is a homeomorphism of X_D^v , the subspace of X_D consisting of $*$ -orderings fully compatible with v , with $X_{D_v} \times \text{hom}(S(\Gamma)/2\Gamma, \{\pm 1\})$, where $\text{hom}(S(\Gamma)/2\Gamma, \{\pm 1\})$ is given the character group topology.*

Extending $$ -orderings.* A major result of Idris establishes a connection between $*$ -orderings and their maximal extensions to extended $*$ -orderings. This connection is extremely valuable for its use in proving theorems about these orderings. See also [C1, Theorem 2.5] for a generalization to arbitrary $*$ -valuations.

Theorem 3.2 [I1, Chapter 2, §3]. *Let P be a $*$ -ordering with order valuation v . Set $Q = \{s + k \mid s \in P, k^* = -k, v(k) > v(s)\}$. Then Q is a maximal extended $*$ -ordering with order valuation v and satisfying $Q \cap S(D) = P$.*

For $*$ -semiorderings, preorderings and fans, extensions can also be defined, but no such beautiful connection with valuation theory is known (cf. [C4, §3]).

c-orderings. Chacron [Ch3] defines a *c-valuation* to be a $*$ -valuation w with a slightly weaker condition for reality (reflecting the fact that c-orderings are not necessarily closed under products); to be precise, $w(a + b) = \min(w(a), w(b))$ for a and b sums of norms and $daa^*d^{-1}(aa^*)^{-1}$ maps to 1 in the residue field for all $a, d \in D^\times$. Order valuations behave much the same as with Baer orderings: the associated ring is total, but not necessarily invariant. Even if it is invariant, the valuation may not be a c-valuation [Ch3]. One of the major results for c-orderings is a characterization of those c-orderings for which the order valuation is a c-valuation.

Theorem 3.3 [Ch3, Theorem 3.1.9]. *Let P be a c -ordering of $(D, *)$. The associated order valuation is a c -valuation if and only if P is a normal c -ordering.*

There are also some important facts known about the structure of finite-dimensional c -valued division rings.

Theorem 3.4 [CW, Theorem 2.1]. *Let D be any finite-dimensional noncommutative division ring with c -valuation v and associated maximal ideal $\mathfrak{m}_v$. If, given any $a \in \mathfrak{m}_v$, there is a $b \in Z_D$ such that $1 + a = b^2$, then D decomposes as a tensor product of $*$ -closed quaternion algebras over Z_D . In particular, this occurs if v restricted to Z_D is Henselian.*

On the other hand, examples can be constructed [CW, §4] to show that not all c -valued division rings must decompose as tensor products of quaternion algebras.

Places. As usual, valuations have associated places; these are defined in [H2, §2] for $*$ -valuations. A place π on a skew field D will be called a $*$ -place if it preserves the involution and satisfies $\pi(d^*d^{-1}) \neq \infty$ for all $d \in D^\times$. These are studied in some detail in [Cr5], where much of [L, §§9–11] is extended to $*$ -places. Just as every ordering of a commutative field has an associated place into the real numbers, $*$ -orderings have associated places into the real quaternions $\mathbb{H}$ by the work of Holland mentioned earlier [H2, §4]. To obtain uniqueness, one must consider two places equivalent if they differ by an automorphism of $\mathbb{H}$. Let $\mathcal{M}(D)$ denote the set of such equivalence classes of $*$ -places. A major result making this set tractable is the following.

Theorem 3.5 [Cr5, Theorem 2.5]. *Elements of $\mathcal{M}(D)$ are determined by their action on $S(D)$.*

The set $\mathcal{M}(D)$ can be given the quotient topology from X_D , making it a compact Hausdorff space, proved much as in the commutative case. As for commutative fields, we define the *real holomorphy ring*

$$\mathcal{H}(D) = \bigcap_{P \in X_D} A(P).$$

The main results known about $\mathcal{H}(D)$ are that it is a (noncommutative) Prüfer domain [C5, Corollary 5.2] and its elements are precisely those $d \in D$ such that there is some positive integer which bounds dd^* with respect to some $*$ -ordering (rather than all $*$ -orderings) [C5, Theorem 5.3].

4. Spaces of orderings and hermitian forms. Baer introduced his notion of ordering precisely because of his interest in hermitian forms. His main theorem in [B, Chap. IV, App. I] is a generalization of Sylvester's theorem of inertia. In spite of this, there has been

far less work done on this topic than on the valuation theory for orderings. The general definitions of hermitian forms and Witt groups can be found in books such as [S], but not the connection with orderings. At this point, we no longer consider c-orderings as their multiplicative properties do not behave well with respect to hermitian forms.

Some brief mentions of hermitian forms in this context are the following. Gross [G] has used Baer orderings in trying to classify infinite-dimensional hermitian spaces. Holland [H2] shows how a positive definite (with respect to a Baer ordering) inner product space over $(D, *)$ induces a positive definite inner product space over the residue field of the order valuation. He also gives a characterization of Hilbert spaces. Piziak [Pi] notes that Baer orderings can sometimes be defined as the set $\{ \Phi(z, z) \mid 0 \neq z \in E \}$, for a hermitian (or more general) space (E, Φ) .

The only effort to carry over the reduced theory of quadratic forms to *-fields, viewing forms as functions on a topological space of orderings, has been by Craven [Cr4], though an involution was introduced into the commutative theory by Knebusch, Rosenberg and Ware [KRW]. A hermitian form over $(D, *)$ can be diagonalized to have the form $\phi = \langle a_1, a_2, \dots, a_n \rangle$, where $a_i \in S(D)^\times$, $i = 1, \dots, n$. We shall denote the Witt group of anisotropic hermitian forms over $(D, *)$ by $W(D, *)$. The basic definitions and properties of hermitian forms over $(D, *)$ and the definition of the Witt group can be found in [S, Chapter 7]. Independent of the question of orderings, there has been some work on extending Springer's Theorem for quadratic forms. In [Ls], Lewis considers the kernel and image of the induced homomorphism $W(F) \rightarrow W(D, *)$, where D is an odd dimensional central simple algebra over a field F and the involution fixes F . He shows that the kernel is not zero in general, as it is in the commutative case, and is able to give quite general conditions under which it is zero.

The involution on D works with the Baer orderings and with the hermitian forms in compatible ways, so that one obtains

Theorem 4.1 [Cr4, Theorem 2.5]. *There exists a bijective correspondence between Baer orderings of $(D, *)$ and group homomorphisms $\sigma : W(D, *) \rightarrow \mathbb{Z}$ satisfying $\sigma(\langle 1 \rangle) = 1$ and $\sigma(\langle a \rangle) = \pm 1$ for all $a \in S(D)^\times$.*

The only situation studied in depth has been that of *-orderings. A canonical factor group of $W(D, *)$, denoted $WR(D, *)$, is defined in [Cr4] on which one can define an associative product $\langle a \rangle \cdot \langle b \rangle = \langle (ab + ba)/2 \rangle$ making $WR(D, *)$ into a ring. We shall use the same notation for elements of this quotient group as for the hermitian forms which represent the equivalence class. The ring homomorphisms of $WR(D, *)$ into $\mathbb{Z}$ correspond to the *-orderings of $(D, *)$, so that there is a "total signature mapping" $WR(D, *) \rightarrow \mathcal{C}(X_D, \mathbb{Z})$, where $\mathcal{C}(X_D, \mathbb{Z})$ is the ring of continuous functions from X_D to the ring of integers with the discrete topology. The main purpose of the paper [Cr4] is to compute the kernel and image of this mapping (actually in the somewhat more general case of an arbitrary preordering in place of $S(D) \cap \Sigma(D)$).

Theorem 4.2 [Cr4, Theorem 6.1]. *The kernel of $WR(D, *) \rightarrow \mathcal{C}(X_D, \mathbb{Z})$ is additively generated by elements of the form*

$$\langle a, -(ad + d^*a) \rangle, \quad a \in S(D)^\times, \quad d \in \Sigma(D).$$

The computation of the image uses the theory of abstract Witt rings and spaces of orderings developed by Marshall [M]. Sections 3, 4, 5, 14, and 15 of [L] are generalized to $*$ -fields in order to prove that the image of $WR(D, *)$ is a space of orderings in this abstract sense. Using the abstract theory, one then obtains a computation of the image, a direct generalization of the theorem of Becker and Bröcker for formally real fields.

Theorem 4.3 [Cr4, Theorem 6.8]. *A function $f \in \mathcal{C}(X_D, \mathbb{Z})$ is in the image of $WR(D, *)$ if and only if for any fan T with X_T finite, we have*

$$\sum_{P \in X_T} f(P) \equiv 0 \pmod{|X_T|}.$$

5. Some open problems.

Does formal reality of D (i.e., $\sum d_i s d_i^* \neq 0$ whenever $s \in S(D)^\times$ and $d_i \in D^\times$) suffice for the existence of a Baer ordering? This is one of the oldest open questions. More generally, what can be said about the intersection of all Baer orderings of a $*$ -field?

As mentioned earlier, in the infinite-dimensional case there are virtually no known results on the structure of ordered $*$ -fields. In particular, there is the (unlikely?) possibility that if D is transcendental over its center, then the order ring $A(P)$ associated with a Baer ordering P must be invariant (see §3, in particular with regard to the results of [MW]).

Unlike the situation for most forms of orderings which can be extended beyond the set of symmetric elements, it is unknown whether every symmetric c-ordering extends to a c-ordering.

Unlike the situation for the Witt ring, there is no known structure theory for the Witt group $W(D, *)$. It should also be interesting to investigate the canonical homomorphism of this group into the ring of continuous functions on the space of Baer orderings (rather than the space of $*$ -orderings). This should be particularly fruitful in the finite-dimensional case.

To apply the ideas in this field to C^* -algebras, one must extend the theory from skew fields to rings with involution. In fact, one would hope that the theory could be extended to arbitrary $*$ -rings, giving a noncommutative version of real algebraic geometry. Initial attempts in this direction do not correspond to the notion of ordering in real algebraic geometry [Sh].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HI 96822

E-mail address: tom@math.hawaii.edu