WHAT IS ALGEBRA?

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1. Introduction

I have considered myself an algebraist at least since getting my Ph.D. in 1973. But this is the first time I have actually been asked to say exactly what algebra is. What I want to convince you of in the next 45 minutes is that the ideas encountered in an algebra course actually underly a whole way of thinking in everyday life, and particularly in approaching problems.

I am not claiming that many people ever use the algorithms taught in an algebra class, but rather that the exposure to those algorithms shapes ones way of thinking and solving problems. This is not an original idea. Indeed,

Thomas Jefferson: "... the faculties of the mind, like the members of the body, are strengthened and improved by exercise. Mathematical reasoning and deductions are, therefore, a fine preparation for investigating the abstruse speculations of the law."

I will present a mixture of history and examples. Jargon: multiple representations, connections, numeracy; [UH system] "symbolic reasoning"

2. History, Definition

algebra – algae-bra (what a mermaid wears) [to get teenagers attention]

ETYMOLOGY: Latin variant of the Arabic al-jabr (wa-l-muqbala), the restoration (and the compensation), addition (and subtraction): al-, the + jabr, bone-setting, restoration (from jabara, to set (bones), restore)

Carl Boyer: "transposition of subtracted terms to the other side of an equation"

Muhammed ben Musa al-Khwarizmi first used the term in 825AD. (His name gives us the word *algorithm*.)

Earlier work by Diophantus appears not to have influenced al-Khwarizmi's work (some controversy among historians), so this seems to be the main influence on European thought – when translated into Latin about 1200. It is based on earlier work by Hindus in the 4th and 5th centuries.

WHY? al-Khwarizmi states that his purpose was to write a popular treatise that would serve the practical ends and needs of the people in their affairs of inheritance and legacies, in their law suits, in trade and commerce, in the surveying of lands and in the digging of canals.

This is very different from the typical reasons for the Greeks, and seems to be the reason he avoided following the Greek tradition, even though one of his coworkers in Baghdad was very involved in translating Greek works into Arabic. This practical handbook was what got the attention of people in Europe! There are three reasons that algebra began to become important to everyday people, not just academicians:

- **Practical:** Hindu-Arabic numerals made computation much easier in commerce than the old Roman numerals.
- **Invention of movable type:** this led to more standardization of symbols.
- **Use in commerce:** It began to be used in Italy in 1200–1300 and spread from there to the rest of Europe.

To make the transition to modern usage, we need to look at just what al-Khwarizmi wrote. It was far from our modern notion of algebra because modern symbols were not invented until after 1500. For example the equal sign was introduced in 1557 by Robert Recorde.

See http://members.aol.com/jeff570/mathsym.html for the history of mathematical symbols.

Example 2.1.

$$x^2 + 21 = 10x$$

was written as

What must be the amount of a square, which when 21 is added to it, becomes equal to the equivalent of 10 roots of that square?

Solutions are $x = 3,\ 7$.

Students in the first grade have a better understanding of symbolic reasoning than that! They can understand something looking like

$$2+? = 5$$

and solve it. I suspect that most fifth graders can continue the series 1, 4, 9, 16,... They are recognizing an algebraic pattern, even though they probably have never seen any formal algebra. In modern usage, algebra is a whole way of looking at concepts symbolically, and this is where is begins to have influence far beyond the algorithms that are taught.

An example: from the newspaper a couple months ago, there was an article that used the letter x as a variable. It has nothing to do with math and I have no memory of what it was about, so I will make one up. "Last night I had a nightmare in which I was giving a lecture in front of x people and I had no clothes on." The value of x is clearly irrelevant to the meaning of the sentence, but it reflects how ingrained the idea of a symbolic variable now is in our culture. A person who has never taken an algebra class would probably never write such a sentence, but would almost certainly understand it.

3. Modern relevance to business

Some quotes from [FS], *Beyond 8th Grade.* The emphasis of the article is not only on technical and problem solving skills, but on quantitative literacy for an informed electorate. Much of the article has to do with what creates success in business: "working smarter, not just working harder" is one of my favorite quotes. It reminds me of something I often tell my students: before jumping into a problem with the first idea that comes to mind, stop and think for minute about other possible solutions and you might find it much easier. An example from high school algebra is to simplify the expression

$$(x-a)(x-b)(x-c)\cdots(x-z)$$

Commission on the skills of the American Workforce 1990: persistence in mathematics is one of the best predictors of success in careers.

Note that this is not talking about technical careers. Any job requires problem solving skills and those skills are enhanced by the learning that takes place in math classes. It is the learning process that shapes the mind's ability to tackle novel problems.

Example 3.1. An example of a modern use of algebra that actually avoids writing down variables, but uses them anyway:

To a mathematician, a spreadsheet is just algebra playing on a popular stage... Figuring out how to translate a task into a spreadsheet design is just like setting up a word problem in algebra; it involves identifying important variables and the relations among them. Preparing a spreadsheet requires equations which are suitably located in the cells. The spreadsheet does the arithmetic and the designer does the algebra.

That is a distinction that I want to emphasize. I distinguish between arithmetic and algebra. Arithmetic is what any calculator does. It is algebra that requires thinking.

4. A hands on example

Two people from Oregon State University, friends who are involved in revising how physics is taught, cornered my wife and I and gave us (separately) the following problem. The idea is to work it out carefully and describe your thought process as you work on it.

For which values of a do the following simultaneous equations have 0,1,2,...,8 solutions:

$$x^2 - y^2 = 0$$
$$(x-a)^2 + y^2 = 1$$

My wife and I approached the problem entirely differently, illustrating here the ideas of multiple representations and connections I mentioned at the beginning of the talk. Take a few minutes to look at it – it will help to have pencil and paper, but it is sufficient to just think about how you would do it as I will not give you time to finish.

Solution 1 View it geometrically. The first equation gives two lines which intersect at the origin $x = \pm y$. The second gives a circle of radius 1 centered at $(a, 0)$. The problem is to determine the number of intersection points for different values of a. Most of the time this is either 4 or 0; when $a = \pm\sqrt{2}$, the circle is tangent to the lines and there are 2 solutions. When $a = \pm 1$, the cirle passes through the origin and there are 3 solutions.

Solution 2 View it algebraically. Substitute x^2 for y^2 in the second equation. You are then solving

$$2x^2 - 2ax + a^2 - 1 = 0.$$

The quadratic formula gives solutions

$$x = \frac{a \pm \sqrt{2 - a^2}}{2}$$

and you can then see the special relations for the values of a above.

References

[B] J. Baumgart, *The History of Algebra*, NCTM, 1969.
[FS] S. Forman and L. A. Steen, *Beyond 8th Grade*, in NCTM 2000 Yearbook.
[V] B.L. van der Waerden, *History of algebra : from al-Khwrizm to Emmy Noether*, Springer, 1985.

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