1. Let $R = \mathbb{C}([0, 1])$ be the ring of continuous real-valued functions on the interval $[0, 1]$, with the usual definitions of sum and product of functions from calculus. Show that $f \in R$ is a zero divisor if and only if $f$ is not identically zero and $\{ x \mid f(x) = 0 \}$ contains an open interval. What are the idempotents of this ring? What are the nilpotents? What are the units?

2. Steinberger, p. 207, #5.


4. Steinberger, p. 211, #15. [$S^1$ is the unit circle, as defined in #6. $SO(2)$, the special orthogonal group, is the group of $2 \times 2$ orthogonal real matrices of determinant 1.]

5. Let $R$ be an integral domain. We say $R$ is a Euclidean ring if there is a function $\alpha \mapsto \| \alpha \|$ from the nonzero elements of $R$ to the nonnegative integers satisfying
   (1) if $\alpha$ is divisible by $\beta$, then $\| \alpha \| \geq \| \beta \|$;
   (2) for all $\alpha, \beta \in R, \beta \neq 0$, there exist $\gamma, \delta \in R$ with $\alpha = \gamma \beta + \delta$ and either $\delta = 0$ or $\| \delta \| < \| \beta \|$.

   a. Prove that every Euclidean ring is a PID.
   b. Let $\alpha, \beta$ be nonzero elements of a Euclidean ring. Define a sequence
      \[
      \begin{align*}
      \alpha &= a_0 \beta + r_1 \\
      \beta &= a_1 r_1 + r_2 \\
      r_1 &= a_2 r_2 + r_3 \\
      & \vdots \\
      r_{n-1} &= a_n r_n
      \end{align*}
      \]
      by applying “division with remainder” until there is no remainder. Prove that you do always stop at some $r_n$, and that $r_n$ is the greatest common divisor of $\alpha$ and $\beta$.
   c. Find the greatest common divisor of 19775 and 18193.

6. a. Let $a, b$ be two elements of a Euclidean domain $R$. Show how, from the Euclidean algorithm, you can find $\lambda, \mu \in R$ with $\lambda a + \mu b = \gcd(a, b)$.
   b. Demonstrate this by finding integers $\lambda, \mu$ such that
      \[235\lambda + 126\mu = 1.\]