BOOLEAN PRODUCT POLYNOMIALS, SCHUR POSITIVITY, AND CHERN PLETHYSM

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ABSTRACT. Let $k \leq n$ be positive integers, and let $X_n = (x_1, \ldots, x_n)$ be a list of n variables. The *Boolean product polynomial* $B_{n,k}(X_n)$ is the product of the linear forms $\sum_{i \in S} x_i$ where S ranges over all k-element subsets of $\{1, 2, \ldots, n\}$. We prove that Boolean product polynomials are Schur positive. We do this via a new method of proving Schur positivity using vector bundles and a symmetric function operation we call *Chern plethysm*. This gives a geometric method for producing a vast array of Schur positive polynomials whose Schur positivity lacks (at present) a combinatorial or representation theoretic proof. We relate the polynomials $B_{n,k}(X_n)$ for certain k to other combinatorial objects including derangements, positroids, alternating sign matrices, and reverse flagged fillings of a partition shape. We also relate $B_{n,n-1}(X_n)$ to a bigraded action of the symmetric group $\mathfrak{S}_n$ on a divergence free quotient of superspace.

1. Introduction

The symmetric group $\mathfrak{S}_n$ of permutations of $[n] := \{1, 2, \ldots, n\}$ acts on the polynomial ring $\mathbb{C}[X_n] := \mathbb{C}[x_1, \ldots, x_n]$ by variable permutation. Elements of the invariant subring

$$\mathbb{C}[X_n]^{\mathfrak{S}_n} := \{F(X_n) \in \mathbb{C}[X_n] \,:\, w.F(X_n) = F(X_n) \text{ for all } w \in \mathfrak{S}_n \,\} \tag{1.1}$$

are called *symmetric polynomials*.

Symmetric polynomials are typically defined using **sums of products** of the variables $x_1, \ldots, x_n$. Examples include the *power sum*, the *elementary symmetric polynomial*, and the *homogeneous symmetric polynomial* which are (respectively)

$$p_k(X_n) = x_1^k + \cdots + x_n^k, \quad e_k(X_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad h_k(X_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}. \tag{1.2}$$

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ with $k \leq n$ parts, we have the *monomial symmetric polynomial*

$$m_\lambda(X_n) = \sum_{i_1, \ldots, i_k \text{ distinct}} x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}, \tag{1.3}$$

as well as the *Schur polynomial* $s_\lambda(X_n)$ whose definition is recalled in Section 2.

Among these symmetric polynomials, the Schur polynomials are the most important. The set of Schur polynomials $s_\lambda(X_n)$ where λ has at most n parts forms a $\mathbb{C}$-basis of $\mathbb{C}[X_n]^{\mathfrak{S}_n}$. A symmetric polynomial $F(X_n)$ is *Schur positive* if its expansion into the Schur basis has nonnegative integer coefficients. Schur positive polynomials admit representation theoretic interpretations involving general linear and symmetric groups as well as geometric interpretations involving cohomology rings of Grassmannians. A central problem in the theory of symmetric polynomials is to decide whether a given symmetric polynomial $F(X_n)$ is Schur positive.

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In addition to the sums of products described above, one can also define symmetric polynomials using **products of sums**. For positive integers $k \leq n$, we define the *Boolean product polynomial*

$$B_{n,k} \coloneqq B_{n,k}(X_n) \coloneqq \prod_{1 \leq i_1 < \cdots < i_k \leq n} (x_{i_1} + \cdots + x_{i_k}). \tag{1.4}$$

For example, when $n = 4$ and $k = 2$, we have

$$B_{4,2}(X_4) = (x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4).$$

One can check $B_{n,1} = x_1 x_2 \cdots x_n = s_{(1^n)}(X_n)$ and $B_{n,2}(X_n) = s_{(n-1,n-2,\ldots,1)}(X_n)$ for $n \geq 2$. We also define a *'total' Boolean product polynomial* $B_n(X_n)$ to be the product of the $B_{n,k}$'s,

$$B_n \coloneqq B_n(X_n) \coloneqq \prod_{k=1}^{n} B_{n,k}(X_n). \tag{1.5}$$

Lou Billera provided our original inspiration for studying $B_n(X_n)$ at a BIRS workshop in 2015. His study of the Boolean product polynomials was partially motivated by the study of the *resonance arrangement*. This is the hyperplane arrangement in $\mathbb{R}^n$ with hyperplanes $\sum_{i \in S} x_i = 0$, where S ranges over all nonempty subsets of $[n]$. The polynomial $B_n(X_n)$ is the defining polynomial of this arrangement. The resonance arrangement is related to double Hurwitz numbers [7], quantum field theory [11], and certain preference rankings in psychology and economics [20]. Enumerating the regions of the resonance arrangement is an open problem. In Section 6, we present further motivation for Boolean product polynomials.

In this paper, we prove that $B_{n,k}(X_n)$ and $B_n(X_n)$ are Schur positive (Theorem 3.8). These results were first announced in [3] and presented at FPSAC 2018. The proof relies on the geometry of vector bundles and involves an operation on symmetric functions and Chern roots which we call *Chern plethysm*. Chern plethysm behaves in some ways like classical plethysm of symmetric functions, but it is clearly a different operation. Our Schur positivity results follow from earlier results of Pragacz [26] and Fulton-Lazarsfeld [14] on numerical positivity in vector bundles over varieties. This method provides a vast array of Schur positive polynomials coming from products of sums, the polynomials $B_{n,k}(X_n)$ and $B_n(X_n)$ among them.

There is a great deal of combinatorial and representation theoretic machinery available for understanding the Schur positivity of sums of products. Schur positive products of sums are much less understood. Despite their innocuous definitions, there is no known combinatorial proof of the Schur positivity of $B_{n,k}(X_n)$ or $B_n(X_n)$, nor is there a realization of these polynomials as the Weyl character of an explicit polynomial representation of GL_n for all k, n. It is the hope of the authors that this paper will motivate further study into Schur positive products of sums.

Toward developing combinatorial interpretations and related representation theory for $B_{n,k}(X_n)$, we study the special cases $B_{n,2}(X_n)$ and $B_{n,n-1}(X_n)$ in more detail. The polynomials $B_{n,2}(X_n)$ are the highest homogeneous component of certain products famously studied by Alain Lascoux [22], namely

$$\prod_{1 \leq i < j \leq n} (1 + x_i + x_j).$$

He showed these polynomials are Schur positive using vector bundles, inspiring the work of Pragacz. It is nontrivial to show the coefficients in his expansion are nonnegative integers. Lascoux's work was also the motivation for the highly influential work of Gessel-Viennot on lattice paths and binomial determinants [16]. In Theorem 4.2, we give the first purely combinatorial interpretation for all of the Schur expansion coefficients in Lascoux's product. Surprisingly, the sum of the coefficients in the Schur expansion of this formula is equal to the number of alternating sign matrices of size n or equivalently the number of totally symmetric self-complementary plane partitions of $2n$ (Corollary 4.4).

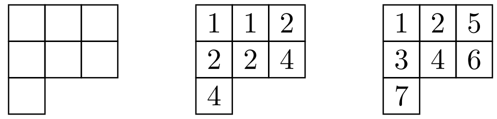


FIGURE 1. The Ferrers diagram of $(3,3,1)$ along with a semistandard and a standard Young tableau of that shape.

In Section 5, we introduce a q-analog of $B_{n,n-1}$. At $q=0$, this polynomial is the Frobenius characteristic of the regular representation of $\mathfrak{S}_n$. At $q=-1$, we get back the Boolean product polynomial. By work of Désarménien-Wachs and Reiner-Webb, we have a combinatorial interpretation of the Schur expansion $B_{n,n-1}$. Furthermore, $B_{n,n-1}$ is the character of a direct sum of Lie_λ representations which has a basis given by derangements in $\mathfrak{S}_n$. At $q=1$, this family of symmetric functions is related to an $\mathfrak{S}_n$-action on positroids. We show the q-analog of $B_{n,n-1}$ is the graded Frobenius characteristic of a bigraded $\mathfrak{S}_n$-action on a divergence free quotient of superspace related to the classical coinvariant algebras, which we call R_n. Following the work of Haglund-Rhoades-Shimozono[19], we extend our construction to the context of ordered set partitions and beyond.

The remainder of the paper is structured as follows. In **Section 2**, we review the combinatorics and representation theory of Schur polynomials. In **Section 3**, we introduce Chern plethysm and explain the relevance of the work of Pragacz to Schur positivity. In **Section 4**, we introduce the reverse flagged fillings in relation to Lascoux's formula in our study of Boolean product polynomials of the form $B_{n,2}$. In **Section 5**, we connect $B_{n,k}$ to combinatorics and representation theory in the special case $k=n-1$. In particular, we introduce a q-deformation of $B_{n,n-1}$ and relate it to the quotient of superspace denoted R_n. We close in **Section 6** with some open problems.

2. Background

We provide a brief introduction to the background and notation we are assuming in this paper. Further details on Schur polynomials and the representation theory of $\mathfrak{S}_n$ and GL_n can be found in [13].

2.1. **Partitions, tableaux, and Schur polynomials.** A *partition of* n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ of positive integers such that $\lambda_1 + \cdots + \lambda_k = n$. We write $\lambda \vdash n$ or $|\lambda| = n$ to mean that λ is a partition of n. We also write $\ell(\lambda) = k$ for the number of parts of λ. The *Ferrers diagram* of a partition λ consists of λ_i left justified boxes in row i. The Ferrers diagram of $(3,3,1) \vdash 7$ is shown on the left in Figure 1. We identify partitions with their Ferrers diagrams throughout.

If λ is a partition, a *semistandard tableau* T of shape λ is a filling $T : \lambda \to \mathbb{Z}_{>0}$ of the boxes of λ with positive integers such that the entries increase weakly across rows and strictly down columns. A semistandard tableau of shape $(3,3,1)$ is shown in the middle of Figure 1. Let $\mathrm{SSYT}(\lambda, \leq n)$ be the family of all semistandard tableaux of shape λ with entries $\leq n$. A semistandard tableau is *standard* if its entries are $1, 2, \ldots, |\lambda|$. A standard tableau, or standard Young tableau, is shown on the right in Figure 1.

Given a semistandard tableau T, define a monomial $x^T := x_1^{m_1(T)} x_2^{m_2(T)} \cdots$, where $m_i(T)$ is the multiplicity of i as an entry in T. In the above example, we have $x^T = x_1^2 x_2^3 x_4^2$. The *Schur polynomial* $s_\lambda(X_n)$ is the corresponding generating function

$$s_\lambda(X_n) := \sum_{T \in \mathrm{SSYT}(\lambda, \leq n)} x^T. \tag{2.1}$$

Observe that $s_\lambda(X_n) = 0$ whenever $\ell(\lambda) > n$.

An alternative formula for the Schur polynomial can be given as a ratio of determinants as follows. Let $\Delta_n := \prod_{1 \leq i < j \leq n}(x_i - x_j) = \sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) \cdot (w.x_1^{n-1}x_2^{n-2}\cdots x_n^0)$ be the Vandermonde determinant. Given any polynomial $f \in \mathbb{C}[X_n]$, define a symmetric polynomial $A_n(f)$ by

$$A_n(f) := \frac{1}{\Delta_n}\sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) \cdot (w.f). \tag{2.2}$$

Let $\mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0)$ be a partition with $\leq n$ parts. The Schur polynomial $s_\mu(X_n)$ can also be obtained applying A_n to the monomial $x_1^{\mu_1+n-1}x_2^{\mu_2+n-2}\cdots x_n^{\mu_n}$:

$$s_\mu(X_n) = A_n(x_1^{\mu_1+n-1}x_2^{\mu_2+n-2}\cdots x_n^{\mu_n}). \tag{2.3}$$

2.2. **GL_n-modules and Weyl characters.** Let GL_n be the group of invertible $n \times n$ complex matrices, and let W be a finite-dimensional representation of GL_n with underlying group homomorphism $\rho : GL_n \to GL(W)$. The *Weyl character* of ρ is the function $\mathrm{ch}_\rho : (\mathbb{C}^\times)^n \to \mathbb{C}$ defined by

$$\mathrm{ch}_\rho(x_1, \ldots, x_n) := \mathrm{trace}(\rho(\mathrm{diag}(x_1, \ldots, x_n))) \tag{2.4}$$

which sends an n-tuple $(x_1, \ldots, x_n)$ of nonzero complex numbers to the trace of the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_n) \in GL_n$ as an operator on W. The function ch_ρ satisfies $\mathrm{ch}_\rho(x_1, \ldots, x_n) = \mathrm{ch}_\rho(x_{w(1)}, \ldots, x_{w(n)})$ for any $w \in \mathfrak{S}_n$.

A representation $\rho : GL_n \to GL(W)$ is *polynomial* if W is finite-dimensional and there exists a basis $\mathcal{B}$ of W such that the entries of the matrix $[\rho(g)]_\mathcal{B}$ representing $\rho(g)$ are polynomial functions of the entries of $g \in GL_n$. This property is independent of the choice of basis $\mathcal{B}$. In this case, the Weyl character $\mathrm{ch}_\rho \in \mathbb{C}[X_n]^{\mathfrak{S}_n}$ is a symmetric polynomial. We will only consider polynomial representations in this paper.

Let $\lambda \vdash d$, and let T be a standard Young tableau with d boxes. Let $R_T, C_T \subseteq \mathfrak{S}_d$ be the subgroups of permutations in $\mathfrak{S}_d$ which stabilize the rows and columns of T, respectively. For the standard tableau of shape $(3,3,1)$ shown in Figure 1, we have $R_T = \mathfrak{S}_{\{1,2,5\}} \times \mathfrak{S}_{\{3,4,6\}} \times \mathfrak{S}_{\{7\}}$ and $C_T = \mathfrak{S}_{\{1,3,7\}} \times \mathfrak{S}_{\{2,4\}} \times \mathfrak{S}_{\{5,6\}}$. The *Young idempotent* $\varepsilon_\lambda \in \mathbb{C}[\mathfrak{S}_d]$ is the group algebra element

$$\varepsilon_\lambda := \sum_{w \in R_T}\sum_{u \in C_T} \mathrm{sign}(u) \cdot uw \in \mathbb{C}[\mathfrak{S}_d]. \tag{2.5}$$

Strictly speaking, the group algebra element ε_λ depends on the standard tableau T, but this dependence is only up to conjugacy by an element of $\mathfrak{S}_d$ and will be ignored.

Let $V = \mathbb{C}^n$ be the standard n-dimensional complex vector space. The symmetric group $\mathfrak{S}_d$ acts on the d-fold tensor product $V \otimes \cdots \otimes V$ *on the right* by permuting tensor factors:

$$(v_1 \otimes \cdots \otimes v_d).w := v_{w^{-1}(1)} \otimes \cdots \otimes v_{w^{-1}(d)}, \quad v_i \in V, w \in \mathfrak{S}_d. \tag{2.6}$$

By linear extension, we have an action of the group algebra $\mathbb{C}[\mathfrak{S}_d]$ on $V \otimes \cdots \otimes V$. If λ is a partition, the *Schur functor* $\mathbb{S}^\lambda(\cdot)$ attached to λ is defined by

$$\mathbb{S}^\lambda(V) := (V \otimes \cdots \otimes V)\varepsilon_\lambda. \tag{2.7}$$

We have $\mathbb{S}^\lambda(V) = 0$ whenever $\ell(\lambda) > \dim(V)$. Two special cases are of interest. If $\lambda = (d)$ is a single row, then $\mathbb{S}^{(d)}(V) = Sym^d V$ is the d^{th} symmetric power. If $\lambda = (1^d)$ is a single column, then $\mathbb{S}^{(1^d)}(V) = \wedge^d V$ is the d^{th} exterior power.

The group $GL_n = GL(V)$ acts on $V \otimes \cdots \otimes V$ *on the left* by the diagonal action

$$g.(v_1 \otimes \cdots \otimes v_d) := (g.v_1) \otimes \cdots \otimes (g.v_d), \quad v_i \in V, \; g \in GL_n. \tag{2.8}$$

This commutes with the action of $\mathbb{C}[\mathfrak{S}_d]$ and so turns $\mathbb{S}^\lambda(V)$ into a GL_n-module. We quote the following bromides of GL_n-representation theory.

(1) If $\ell(\lambda) \leq n$, the module $\mathbb{S}^\lambda(V)$ is an irreducible polynomial representation of GL_n with Weyl character given by the Schur polynomial $s_\lambda(X_n)$.
(2) The modules $\mathbb{S}^\lambda(V)$ for $\ell(\lambda) \leq n$ form a complete list of the nonisomorphic irreducible polynomial representations of GL_n.
(3) Any polynomial GL_n-representation may be expressed uniquely as a direct sum of the modules $\mathbb{S}^\lambda(V)$.

A symmetric polynomial $f(X_n) \in \mathbb{C}[X_n]^{\mathfrak{S}_n}$ is therefore Schur positive if and only if it is the Weyl character of a polynomial representation of GL_n.

2.3. **$\mathfrak{S}_n$-modules and Frobenius image.** Let $X = (x_1, x_2, \dots)$ be an infinite list of variables. For $d > 0$, the *power sum symmetric function* is $p_d(X) = x_1^d + x_2^d + \cdots$; this is an element of the ring $\mathbb{C}[[X]]$ of formal power series in X. The *ring of symmetric functions*

$$\Lambda := \mathbb{C}[p_1(X), p_2(X), \dots] \tag{2.9}$$

is the $\mathbb{C}$-subalgebra of $\mathbb{C}[[X]]$ freely generated by the $p_d(X)$. The algebra Λ is graded; let Λ_n be the subspace of homogeneous degree n symmetric functions so that $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$.

For $d \geq 0$, the *elementary symmetric function* is $e_d(X) := \sum_{1 \leq i_1 < i_2 < \cdots < i_d} x_{i_1} x_{i_2} \cdots x_{i_d}$, and the *homogeneous symmetric function* is $h_d(X) := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d} x_{i_1} x_{i_2} \cdots x_{i_d}$. If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition, we define

$$p_\lambda(X) := p_{\lambda_1}(X) p_{\lambda_2}(X) \cdots, \quad e_\lambda(X) := e_{\lambda_1}(X) e_{\lambda_2}(X) \cdots, \quad \text{and} \quad h_\lambda(X) := h_{\lambda_1}(X) h_{\lambda_2}(X) \cdots$$

Given a partition λ, the *Schur function* $s_\lambda(X) \in \Lambda$ is the formal power series $s_\lambda(X) := \sum_T x^T$, where T ranges over all semistandard tableaux of shape λ. The set $\{s_\lambda(X) : \lambda \vdash n\}$ forms a basis for Λ_n as a $\mathbb{C}$-vector space.

The irreducible $\mathfrak{S}_n$-modules are naturally indexed by partitions of n. If $\lambda \vdash n$, let $S^\lambda = \mathbb{C}[\mathfrak{S}_n]\varepsilon_\lambda$ be the corresponding irreducible module. If V is any $\mathfrak{S}_n$-module, there exist unique integers $m_\lambda \geq 0$ such that $V \cong \bigoplus_{\lambda \vdash n} m_\lambda S^\lambda$. The *Frobenius image* $\mathrm{Frob}(V) \in \Lambda_n$ is given by $\mathrm{Frob}(V) := \sum_{\lambda \vdash n} m_\lambda s_\lambda(X)$. A symmetric function $F(X) \in \Lambda_n$ is therefore Schur positive if and only if $F(X)$ is the Frobenius image of some $\mathfrak{S}_n$-module V.

Let V be an $\mathfrak{S}_n$-module, and let W be an $\mathfrak{S}_m$-module. The tensor product $V \otimes W$ is naturally an $\mathfrak{S}_n \times \mathfrak{S}_m$-module. We have an embedding $\mathfrak{S}_n \times \mathfrak{S}_m \subseteq \mathfrak{S}_{n+m}$ by letting $\mathfrak{S}_n$ act on the first n letters and letting $\mathfrak{S}_m$ act on the last m letters. The *induction product* of V and W is

$$V \circ W := (V \otimes W) \uparrow_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} . \tag{2.10}$$

The effect of induction product on Frobenius image is

$$\mathrm{Frob}(V \circ W) = \mathrm{Frob}(V) \cdot \mathrm{Frob}(W). \tag{2.11}$$

Suppose $V = \bigoplus_{i \geq 0} V_i$ is a graded $\mathfrak{S}_n$-module such that each component V_i is finite-dimensional. The *graded Frobenius image* is

$$\mathrm{grFrob}(V; t) := \sum_{i \geq 0} \mathrm{Frob}(V_i) \cdot t^i. \tag{2.12}$$

More generally, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded $\mathfrak{S}_n$-module with each component $V_{i,j}$ finite-dimensional, the *bigraded Frobenius image* is

$$\mathrm{grFrob}(V; t, q) := \sum_{i,j \geq 0} \mathrm{Frob}(V_{i,j}) \cdot t^i q^j. \tag{2.13}$$

2.4. **The coinvariant algebra.** Let $\mathbb{C}[X_n]_+^{\mathfrak{S}_n} \subseteq \mathbb{C}[X_n]$ be the vector space of symmetric polynomials with vanishing constant term, and let $\langle \mathbb{C}[X_n]_+^{\mathfrak{S}_n} \rangle \subseteq \mathbb{C}[X_n]$ be the ideal generated by this space. We have the generating set $\langle \mathbb{C}[X_n]_+^{\mathfrak{S}_n} \rangle = \langle e_1(X_n), e_2(X_n), \ldots, e_n(X_n) \rangle$. The *coinvariant ring* is the graded $\mathfrak{S}_n$-module

$$\mathbb{C}[X_n]/\langle \mathbb{C}[X_n]_+^{\mathfrak{S}_n} \rangle = \mathbb{C}[X_n]/\langle e_1(X_n), e_2(X_n), \ldots, e_n(X_n) \rangle. \tag{2.14}$$

As an ungraded $\mathfrak{S}_n$-module, the coinvariant ring is isomorphic to the regular representation $\mathbb{C}[\mathfrak{S}_n]$. The graded $\mathfrak{S}_n$-module structure of the coinvariant ring is governed by the combinatorics of tableaux.

Let $\mathrm{SYT}(n)$ denote the set of all standard Young tableaux with n boxes (of any partition shape). We let $\mathrm{shape}(T)$ be the partition obtained by erasing the entries of the tableau T. If $T \in \mathrm{SYT}(n)$, an element $1 \leq i \leq n-1$ is a *descent* if $i+1$ appears in a strictly lower row than i in T. Otherwise, i is an *ascent* of T. The *major index* $\mathrm{maj}(T)$ is the sum of the descents in T. For example, the standard tableau on the right in Figure 1 has descents at $2, 5$, and 6 so its major index is 13. The following graded Frobenius image of the coinvariant ring indexed by standard Young tableaux is due to Lusztig (unpublished) and Stanley [32]. See also [35].

Theorem 2.1 (Lusztig–Stanley, [32, Prop. 4.11])**.** *For any positive integer n, we have*

$$\mathrm{grFrob}(\mathbb{C}[X_n]/\langle \mathbb{C}[X_n]_+^{\mathfrak{S}_n} \rangle; t) = \sum_{T \in \mathrm{SYT}(n)} t^{\mathrm{maj}(T)} \cdot s_{\mathrm{shape}(T)}(X).$$

3. Chern plethysm and Schur positivity

3.1. **Chern classes and Chern roots.** We describe basic properties of vector bundles and their Chern roots from a combinatorial point of view; see [12] for more details. No knowledge of geometry is necessary to understand the new results of this paper, but we do rely on the geometric results from the liturature for some of our proofs and motivation.

Let X be a smooth complex projective variety, and let $H^\bullet(X)$ be the singular cohomology of X with integer coefficients. Let $\mathcal{E} \twoheadrightarrow X$ be a complex vector bundle over X of rank n. For any point $p \in X$, the fiber $\mathcal{E}_p$ of $\mathcal{E}$ over p is an n-dimensional complex vector space. For $1 \leq i \leq n$, we have the *Chern class* $c_i(\mathcal{E}) \in H^{2i}(X)$. The sum of these Chern classes inside $H^\bullet(X)$ is the *total Chern class* $c_\bullet(\mathcal{E}) := 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots + c_n(\mathcal{E})$.

If $\mathcal{E} \twoheadrightarrow X$ and $\mathcal{F} \twoheadrightarrow X$ are two vector bundles, we can form their *direct sum bundle* (or *Whitney sum*) by the rule $(\mathcal{E} \oplus \mathcal{F})_p := \mathcal{E}_p \oplus \mathcal{F}_p$ for all $p \in X$. The ranks of these bundles are related by $\mathrm{rank}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{rank}(\mathcal{E}) + \mathrm{rank}(\mathcal{F})$. The *Whitney sum formula* [12, Thm. 3.2e] states that the corresponding total Chern classes are related by $c_\bullet(\mathcal{E} \oplus \mathcal{F}) = c_\bullet(\mathcal{E}) \cdot c_\bullet(\mathcal{F})$.

Recall that a *line bundle* is a vector bundle of rank 1. Let $\mathcal{E} \twoheadrightarrow X$ be a rank n vector bundle. If we can express $\mathcal{E}$ as a direct sum of n line bundles, i.e. $\mathcal{E} = \ell_1 \oplus \cdots \oplus \ell_n$, then the Whitney sum formula guarantees that the total Chern class of $\mathcal{E}$ factors as

$$c_\bullet(\mathcal{E}) = (1 + x_1) \cdots (1 + x_n) \tag{3.1}$$

where $x_i = c_1(\ell_i) \in H^2(X)$. Despite the fact $\mathcal{E}$ is not necessarily a direct sum of line bundles, by the *splitting principle* [12, Rmk. 3.2.3] there exists,

- a space X' with a map $X' \to X$ such that the induced cohomology map $H^\bullet(X) \to H^\bullet(X')$ is an injection, and
- elements $x_1, \ldots, x_n \in H^2(X')$ such that the factorization (3.1) holds.

The elements $x_1, \ldots, x_n$ of the second cohomology group of X' are the *Chern roots* of the bundle $\mathcal{E}$. Any symmetric polynomial in $x_1, \ldots, x_n$ lies in the cohomology ring $H^\bullet(X)$ of X.

3.2. **Chern plethysm.** Let $\mathcal{E} \twoheadrightarrow X$ be a rank n complex vector bundle over a smooth algebraic variety, and let $x_1, \ldots, x_n$ be the Chern roots of $\mathcal{E}$. If $F \in \Lambda$ is a symmetric function, we define the *Chern plethysm* $F(\mathcal{E})$ to be the result of plugging the Chern roots $x_1, \ldots, x_n$ of $\mathcal{E}$ into n of the arguments of F, and setting all other arguments of F equal to zero.

Informally, the expression $F(\mathcal{E})$ evaluates F at the Chern roots of $\mathcal{E}$. If $F \in \Lambda_d$ is homogeneous of degree d, then $F(\mathcal{E}) = F(x_1, \ldots, x_n)$ is a polynomial of degree d in the x_i. Observe, the degree d is independent of the rank n of $\mathcal{E}$. Although $x_1, \ldots, x_n$ themselves may not lie in $H^\bullet(X)$, the symmetry of F guarantees that $F(\mathcal{E}) = F(x_1, \ldots, x_n) \in H^{2d}(X)$. The x_i appearing in $F(\mathcal{E})$ may be viewed as formal variables; this is justified by the following remarks.

Remark 3.1. *The reader may worry that, since $F(\mathcal{E})$ lies in the cohomology ring $H^\bullet(X)$ of the base space X of the bundle $\mathcal{E}$, relations in $H^\bullet(X)$ may preclude the use of Chern plethysm in proving that polynomials in the ring $\mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ are Schur positive. Fortunately, the base space X may be chosen so that the Chern roots $x_1, \ldots, x_n$ are algebraically independent.*

One can take X to be the n-fold product $\mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty$ of infinite-dimensional complex projective space with itself, and let $\mathcal{E} = \ell_1 \oplus \cdots \oplus \ell_n$ be the direct sum of the tautological line bundles over the n factors of X. The Chern roots of $\mathcal{E}$ are the variables $x_1, \ldots, x_n$ in the presentation $H^\bullet(X) = \mathbb{Z}[x_1, \ldots, x_n]$. For this reason, there is no harm done in thinking of $F(\mathcal{E})$ as an honest symmetric polynomial in $\mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$.

Remark 3.2. *In the combinatorial view of symmetric functions, the x_i are regarded as formal symbols, independent of algebraic or geometric interpretations. We keep the status of x_i as a Chern root more explicit, e.g. writing $F(\mathcal{E})$ instead of $F(x_1, \ldots, x_n)$. Although we hope that the combinatorics of Chern plethysm will be better understood, we do this because (1) Chern plethysm can be unstable as the number of Chern roots tends to infinity – see Proposition 3.4 – and (2) the proofs of our Schur positivity results rely on the geometry of vector bundles.*

If $\mathcal{E}$ is a rank n vector bundle and $1 \leq k \leq n$, the Chern plethysm $e_k(\mathcal{E})$ has a geometric interpretation as the k^{th} Chern class of $\mathcal{E}$,

$$e_k(\mathcal{E}) = c_k(\mathcal{E}) \in H^{2k}(X). \tag{3.2}$$

If λ is any partition, this yields a geometric interpretation of $e_\lambda(\mathcal{E})$ as a product of Chern classes in $H^\bullet(X)$. It may be interesting to find geometric interpretations of other Chern plethysms $F(\mathcal{E})$.

Performing fiberwise operations on vector bundles induces linear changes in their Chern roots. We give three examples of how this applies to Chern plethysm. Let $\mathcal{E}$ be a vector bundle with Chern roots $x_1, \ldots, x_n$, and let $\mathcal{F}$ be a vector bundle with Chern roots $y_1, \ldots, y_m$.

- By the Whitney sum formula, the Chern roots of the direct sum $\mathcal{E} \oplus \mathcal{F}$ are the multiset union of $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ so that

$$F(\mathcal{E} \oplus \mathcal{F}) = F(x_1, \ldots, x_n, y_1, \ldots, y_m). \tag{3.3}$$

- The tensor product bundle $\mathcal{E} \otimes \mathcal{F}$ is defined by $(\mathcal{E} \otimes \mathcal{F})_p := \mathcal{E}_p \otimes \mathcal{F}_p$ for all $p \in X$. The Chern roots of $\mathcal{E} \otimes \mathcal{F}$ are the multiset of sums $x_i + y_j$ where $1 \leq i \leq n$ and $1 \leq j \leq m$ so that

$$F(\mathcal{E} \otimes \mathcal{F}) = F(\overbrace{\ldots, x_i + y_j, \ldots}^{1 \leq i \leq n,\ 1 \leq j \leq m}). \tag{3.4}$$

  Since F is symmetric, the ordering of these arguments is immaterial.
- Let λ be a partition. We may apply the Schur functor $\mathbb{S}^\lambda$ to the bundle $\mathcal{E}$ to obtain a new bundle $\mathbb{S}^\lambda(\mathcal{E})$ with fibers $\mathbb{S}^\lambda(\mathcal{E})_p := \mathbb{S}^\lambda(\mathcal{E}_p)$. The Chern roots of $\mathbb{S}^\lambda(\mathcal{E})$ are the multiset of

sums $\sum_{\square\in\lambda} x_{T(\square)}$ where T varies over $\mathrm{SSYT}(\lambda, \leq n)$ so that

$$F(\mathbb{S}^\lambda(\mathcal{E})) = F(\overbrace{\ldots, \sum_{\square\in\lambda} x_{T(\square)}, \ldots}^{T\in\mathrm{SSYT}(\lambda,\leq n)}). \tag{3.5}$$

For example, if $\lambda = (2,1)$ and $n = 3$, the elements of $\mathrm{SSYT}(\lambda, \leq n)$ are

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}$$

and $F(\mathbb{S}^\lambda(\mathcal{E}))$ is the expression

$$F(2x_1 + x_2, x_1 + 2x_2, 2x_1 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3, 2x_2 + x_3, x_1 + 2x_3, 2x_2 + x_3).$$

As before, the symmetry of F makes the order of substitution irrelevant.

If $F, G \in \Lambda$ are any symmetric functions and $\alpha, \beta \in \mathbb{C}$ are scalars, we have the laws of polynomial evaluation

$$\begin{cases} (F \cdot G)(\mathcal{E}) = F(\mathcal{E}) \cdot G(\mathcal{E}), \\ (\alpha F + \beta G)(\mathcal{E}) = \alpha F(\mathcal{E}) + \beta G(\mathcal{E}), \\ \alpha(\mathcal{E}) = \alpha \end{cases} \tag{3.6}$$

for any vector bundle $\mathcal{E}$.

3.3. **Comparison with classical plethysm.** Given a symmetric function $F \in \Lambda$ and any rational function $E = E(t_1, t_2, \ldots)$ in a countable set of variables, there is a classical notion of plethysm $F[E]$. The quantity $F[E]$ is determined by imposing the same relations as (3.6), i.e.

$$\begin{cases} (F \cdot G)[E] = F[E] \cdot G[E], \\ (\alpha F + \beta G)[E] = \alpha F[E] + \beta G[E], \\ \alpha[E] = \alpha \end{cases} \tag{3.7}$$

for all $F, G \in \Lambda$ and $\alpha, \beta \in \mathbb{C}$ together with the condition

$$p_k[E] = p_k[E(t_1, t_2, \ldots)] := E(t_1^k, t_2^k, \ldots), \quad k \geq 1. \tag{3.8}$$

Since the power sums $p_1, p_2, \ldots$ freely generate Λ as a $\mathbb{C}$-algebra, this defines $F[E]$ uniquely. For more information on classical plethysm, see [18].

Let us compare and contrast the two notions of plethysm $F(\mathcal{E})$ and $F[E]$. Some simple observations are the following.

(1) For any bundle $\mathcal{E}$, the degree of the polynomial $F(\mathcal{E})$ equals the degree $\deg(F)$ of F. However, if E is a polynomial (or formal power series) of degree e, the degree of $F[E]$ is $e \cdot \deg(F)$.

(2) If $x_1, \ldots, x_n$ are the Chern roots of $\mathcal{E}$ we have the relation

$$F(\mathcal{E}) = F[x_1 + \cdots + x_n] = F[X_n] \tag{3.9}$$

for any symmetric function F, where we adopt the plethystic shorthand $X_n = x_1 + \cdots + x_n$ for a sum over an alphabet of n variables. The direct sum operation on vector bundles corresponds to classical plethystic sum in the sense that if $\mathcal{F}$ is another vector bundle with Chern roots $y_1, \ldots, y_m$, then

$$F(\mathcal{E} \oplus \mathcal{F}) = F[x_1 + \cdots + x_n + y_1 + \cdots + y_m] = F[X_n + Y_m]. \tag{3.10}$$

(3) There is no natural interpretation of $F(\mathcal{E} \otimes \mathcal{F})$ or $F(\mathbb{S}^\lambda(\mathcal{E}))$ in terms of classical plethysm.

(4) There does not seem to be a natural interpretation of expressions like $F[X_n \cdot Y_m] = F(\ldots, x_i y_j, \ldots)$ in terms of Chern plethysm.

Equation (3.10) allows us to expand Chern plethysms of direct sums $\mathcal{E} \oplus \mathcal{F}$ of bundles. The proof of the following proposition is left to the reader.

Proposition 3.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles, let $k \geq 0$, and let ν be a partition. We have*

$$p_k(\mathcal{E} \oplus \mathcal{F}) = p_k(\mathcal{E}) + p_k(\mathcal{F}) \tag{3.11}$$

$$e_k(\mathcal{E} \oplus \mathcal{F}) = \sum_{r=0}^{k} e_r(\mathcal{E}) e_{k-r}(\mathcal{F}) \tag{3.12}$$

$$h_k(\mathcal{E} \oplus \mathcal{F}) = \sum_{r=0}^{k} h_r(\mathcal{E}) h_{k-r}(\mathcal{F}) \tag{3.13}$$

$$s_\nu(\mathcal{E} \oplus \mathcal{F}) = \sum_{\lambda,\mu} c^{\nu}_{\lambda,\mu} s_\lambda(\mathcal{E}) s_\mu(\mathcal{F}) \tag{3.14}$$

where $c^{\nu}_{\lambda,\mu}$ is a Littlewood-Richardson coefficient.

The Chern plethystic expansions of tensor products $\mathcal{E} \otimes \mathcal{F}$ is more complicated. As with classical plethysm, it is easiest to calculate Chern plethysms involving power sums.

Proposition 3.4. *Let $\mathcal{E}$ be a vector bundle of rank n, and let $\mathcal{F}$ be a vector bundle of rank m. We have*

$$p_k(\mathcal{E} \otimes \mathcal{F}) = m \cdot p_k(\mathcal{E}) + n \cdot p_k(\mathcal{F}) + \sum_{r=1}^{k-1} \binom{k}{r} \cdot p_r(\mathcal{E}) p_{k-r}(\mathcal{F}). \tag{3.15}$$

Proof. Let $x_1, \ldots, x_n$ be the Chern roots of $\mathcal{E}$, and let $y_1, \ldots, y_m$ be the Chern roots of $\mathcal{F}$. The proposition is equivalent to the polynomial identity

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (x_i + y_j)^k = m \cdot \sum_{i=1}^{n} x_i^k + n \cdot \sum_{j=1}^{m} y_j^k + \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \sum_{r=1}^{k-1} \binom{k}{r} \cdot x_i^r y_j^{k-r}. \tag{3.16}$$

□

Proposition 3.4 reveals another difference between Chern and classical plethysm. For any symmetric functions $F, G \in \Lambda$ in infinitely many variables $x_1, x_2, \ldots$, the classical plethysm $F[G] \in \Lambda$ is a well-defined symmetric function. However, the right-hand-side of Equation (3.15) diverges when $n \to \infty$ or $m \to \infty$. Chern plethysms $F(\mathcal{E} \otimes \mathcal{F})$ of tensor products 'remember' the ranks of $\mathcal{E}$ and $\mathcal{F}$, so it does not in general make sense to extend them to an infinite variable set.

Equation (3.6) can be used to prove associativity and linearity properties of $F(\mathcal{E})$ involving the outer argument F. Similar relations involving the inner argument $\mathcal{E}$ follow from properties of $\oplus$ and $\otimes$ on vector spaces/bundles can be determined as follows.

Proposition 3.5. *Let F be a symmetric function, and let $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ be vector bundles. We have the associativity relations*

$$F((\mathcal{E} \oplus \mathcal{F}) \oplus \mathcal{G}) = F(\mathcal{E} \oplus (\mathcal{F} \oplus \mathcal{G})) \quad \text{and} \quad F((\mathcal{E} \otimes \mathcal{F}) \otimes \mathcal{G}) = F(\mathcal{E} \otimes (\mathcal{F} \otimes \mathcal{G})), \tag{3.17}$$

the commutativity relations

$$F(\mathcal{E} \oplus \mathcal{F}) = F(\mathcal{F} \oplus \mathcal{E}) \quad \text{and} \quad F(\mathcal{E} \otimes \mathcal{F}) = F(\mathcal{F} \otimes \mathcal{E}), \tag{3.18}$$

and the distributivity relation

$$F(\mathcal{E} \otimes (\mathcal{F} \oplus \mathcal{G})) = F((\mathcal{E} \otimes \mathcal{F}) \oplus (\mathcal{E} \otimes \mathcal{G})). \tag{3.19}$$

Proof. This follows from the description of the Chern roots of a direct sum or tensor product of vector bundles in terms of the Chern roots of the constituent bundles. □

For our final comparison of Chern and classical plethysms, we consider the involution ω of Λ which interchanges $e_k(X)$ and $h_k(X)$. The image ωF of a symmetric function $F \in \Lambda$ under ω may be expressed using classical plethysm as follows. For $F \in \Lambda$ and $X = x_1 + x_2 + \cdots$, let $F[\epsilon X]$ be the result of replacing each x_i in F with $-x_i$. In general, the symmetric functions $F[\epsilon X]$ and $F[-X]$ are **not** equal. However, we have

$$\omega F[X] = F[-\epsilon X] \tag{3.20}$$

for any symmetric function F. Equation (3.20) follows from $\omega p_k(X) = (-1)^{k-1} p_k(X)$ for all $k \geq 0$. See [18, Ex. 1.26] for more details on the operator ϵ.

There does not appear to be a direct connection between Chern plethysm and ω. On the other hand, the operator ϵ arises naturally in Chern plethystic calculus.

Proposition 3.6. *Let $F[X] \in \Lambda$, and let $G[X] = F[\epsilon X]$. Let $\mathcal{E}$ be a vector bundle over X, and let $\mathcal{E}^*$ be the* dual bundle *whose fiber over a point $p \in X$ is $\mathcal{E}_p^* = \mathrm{Hom}_{\mathbb{C}}(\mathcal{E}_p, \mathbb{C})$. We have*

$$F(\mathcal{E}^*) = G(\mathcal{E}). \tag{3.21}$$

Proof. If the Chern roots of $\mathcal{E}$ are $x_1, \ldots, x_n$, the Chern roots of $\mathcal{E}^*$ are $-x_1, \ldots, -x_n$. □

Classical plethystic calculus is among the most powerful tools in symmetric function theory (see e.g. [6]). In this paper, we will use geometric results to prove the Schur positivity of polynomials coming from Chern plethysm. It is our hope that Chern plethystic calculus will prove useful in the future.

3.4. **Chern plethysm and Schur positivity.** We have the following positivity result of Pragacz, stated in the language of Chern plethysm.

Theorem 3.7. *(Pragacz* [27, Cor. 7.2]*, see also* [26, p. 34]*) Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be vector bundles, and let $\lambda, \mu^{(1)}, \ldots, \mu^{(k)}$ be partitions. There exist nonnegative integers $c_{\nu^{(1)},\ldots,\nu^{(k)}}^{\lambda,\mu^{(1)},\ldots,\mu^{(k)}}$ so that*

$$s_\lambda(\mathbb{S}^{\mu^{(1)}}(\mathcal{E}_1) \otimes \cdots \otimes \mathbb{S}^{\mu^{(k)}}(\mathcal{E}_k)) = \sum_{\nu^{(1)},\ldots,\nu^{(k)}} c_{\nu^{(1)},\ldots,\nu^{(k)}}^{\lambda,\mu^{(1)},\ldots,\mu^{(k)}} \cdot s_{\nu^{(1)}}(\mathcal{E}_1) \cdots s_{\nu^{(k)}}(\mathcal{E}_k).$$

Pragacz's Theorem 3.7 relies on deep work of Fulton and Lazarsfeld [14] in the context of numerical positivity. The Hard Lefschetz Theorem is a key tool in [14].

We are ready to deduce the Schur positivity of the Boolean product polynomials.

Theorem 3.8. *The Boolean product polynomials $B_{n,k}(X_n)$ and $B_n(X_n)$ are Schur positive.*

Proof. Let $\mathcal{E} \twoheadrightarrow X$ be a rank n vector bundle over a smooth variety X with Chern roots $x_1, \ldots, x_n$. The k^{th} exterior power $\wedge^k \mathcal{E}$ has Chern roots $\{x_{i_1} + \cdots + x_{i_k} \,:\, 1 \leq i_1 < \cdots < i_k \leq n\}$. By Pragacz's Theorem 3.7, the polynomial $s_\lambda(\wedge^k \mathcal{E})$ is Schur positive for any partition λ. In particular, if we take $\lambda = (1, \ldots, 1)$ to be a single column of size $\binom{n}{k}$, we have

$$s_\lambda\left(\wedge^k \mathcal{E}\right) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} (x_{i_1} + \cdots + x_{i_k}) = B_{n,k}(X_n), \tag{3.22}$$

so $B_{n,k}(X_n)$ has a Schur positive expansion. Since $B_n(X_n) = \prod_{1 \leq k \leq n} B_{n,k}(X_n)$, the Littlewood-Richardson rule implies that $B_n(X_n)$ is also Schur positive. □

As stated in the introduction, it is easy to determine the Schur expansion explicitly for $B_{n,1} = s_{(1^n)}(X_n)$, $B_{n,n} = s_{(1)}(X_n)$, and $B_{n,2} = s_{(n-1,n-2,\ldots,1)}(X_n)$ for $n \geq 2$. We will discuss the Schur expansion for $B_{n,n-1}(X_n)$ in Section 5. No effective formula of the Schur expansion of $B_{n,k}(X_n)$ is known for $3 \leq k \leq n-2$. These Schur expansions (and Schur expansions of Chern plethysms more generally) can be quite complicated. For example, the polynomial $B_{5,3}(X_5)$ has Schur expansion

$$6s_{32221} + 9s_{33211} + 3s_{3322} + 3s_{3331} + 9s_{42211} + 3s_{4222} + 6s_{43111} + 9s_{4321} + 3s_{433} + 3s_{4411} + 3s_{442} + 4s_{52111} + 4s_{5221} + 4s_{5311} + 4s_{532} + 2s_{541} + s_{61111} + s_{6211} + s_{622} + s_{631}.$$

Problem 3.9. *Find a combinatorial interpretation for the coefficients in the Schur expansions of $B_{n,k}(X_n)$ and $B_n(X_n)$.*

Theorem 3.8 guarantees the existence of GL_n-modules whose Weyl characters are $B_{n,k}(X_n)$ and $B_n(X_n)$.

Problem 3.10. *Find natural GL_n-modules $V_{n,k}$ and V_n such that $\mathrm{ch}(V_{n,k}) = B_{n,k}(X_n)$ and $\mathrm{ch}(V_n) = B_n(X_n)$.*

If U and W are GL_n-modules and we endow $U \otimes W$ with the diagonal action $g.(u \otimes w) := (g.u) \otimes (g.w)$ of GL_n, then $\mathrm{ch}(U \otimes W) = \mathrm{ch}(U) \cdot \mathrm{ch}(W)$. If we can find a module $V_{n,k}$ as in Problem 3.10, we can therefore take $V_n = V_{n,1} \otimes V_{n,2} \otimes \cdots \otimes V_{n,n}$.

If $V_{n,k}$ is a GL_n-module as in Problem 3.10, then we must have

$$\dim(V_{n,k}) = B_{n,k}(x_1, \ldots, x_n) \mid_{x_1 = \cdots = x_n = 1} = k^{\binom{n}{k}}. \tag{3.23}$$

A natural vector space of this dimension may be obtained as follows. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{C}^n$. For any subset $I \subseteq [n]$, let $\mathbb{C}^I$ be the span of $\{\mathbf{e}_i : i \in I\}$. Then

$$V_{n,k} := \bigotimes_{I \subseteq [n],\ |I| = k} \mathbb{C}^I \tag{3.24}$$

is a vector space of the correct dimension. The action of the diagonal subgroup $(\mathbb{C}^\times)^n \subseteq GL_n$ preserves each tensor factor of $V_{n,k}$, and we have

$$\mathrm{trace}_{V_{n,k}}(\mathrm{diag}(x_1, \ldots, x_n)) = B_{n,k}(x_1, \ldots, x_n). \tag{3.25}$$

One way to solve Problem 3.10 would be to extend this action to the full general linear group GL_n.

3.5. **Bivariate Boolean Product Polynomials.** What happens when we apply Pragacz's Theorem 3.7 to the case of more than one vector bundle $\mathcal{E}_i$? This yields Schur positivity results involving polynomials over more than one set of variables. For clarity, we describe the case of two bundles here.

Let $\mathcal{E}$ be a vector bundle with Chern roots $x_1, \ldots, x_n$ and $\mathcal{F}$ be a vector bundle with Chern roots $y_1, \ldots, y_m$. For $1 \leq k \leq n$ and $1 \leq \ell \leq m$, we have the extension of the Boolean product polynomial to two sets of variables

$$P_{k,\ell}(X_n; Y_m) := \prod_{1 \leq i_1 < \cdots < i_k \leq n} \ \prod_{1 \leq j_1 < \cdots < j_\ell \leq m} (x_{i_1} + \cdots + x_{i_k} + y_{j_1} + \cdots + y_{j_\ell}). \tag{3.26}$$

Observe that $P_{k,\ell}$ equals the Chern plethysm $e_d(\wedge^k \mathcal{E} \otimes \wedge^\ell \mathcal{F})$, where $d = \binom{n}{k}\binom{m}{\ell}$. By Theorem 3.7, there are nonnegative integers $a_{\lambda,\mu}$ such that

$$P_{j,\ell}(X_n; Y_m) = \sum_{\lambda,\mu} a_{\lambda,\mu} \cdot s_\lambda(X_n) \cdot s_\mu(Y_m). \tag{3.27}$$

Setting the y-variables equal to zero recovers Theorem 3.8.

Equation (3.27) is reminiscent of the *dual Cauchy identity* which uses the Robinson-Schensted-Knuth correspondence to give a combinatorial proof that

$$\prod_{1 \leq i \leq n} \prod_{1 \leq j \leq m} (x_i + y_j) = \sum_{\lambda \subseteq (m^n)} s_\lambda(X_n) \cdot s_{\tilde{\lambda}}(Y_m), \tag{3.28}$$

where $\tilde{\lambda}$ is the transpose of the complement of λ inside the rectangular Ferrers shape (m^n). This raises the following natural problem.

Problem 3.11. *Develop a variant of the RSK correspondence which proves the integrality and nonnegativity of the $a_{\lambda,\mu}$ in Equation (3.27).*

4. A combinatorial interpretation of Lascoux's Formula

Pragacz's Theorem has the following sharpening due to Lascoux in the case of one vector bundle. In fact, Lascoux's Theorem was part of the inspiration for Pragacz's Theorem.

Theorem 4.1. *(Lascoux [22]) Let $\mathcal{E}$ be a rank n vector bundle with Chern roots $x_1, \ldots, x_n$, so that we have the total Chern classes*

$$c(\wedge^2 \mathcal{E}) = \prod_{1 \leq i < j \leq n} (1 + x_i + x_j), \text{ and}$$
$$c(Sym^2 \mathcal{E}) = \prod_{1 \leq i \leq j \leq n} (1 + x_i + x_j).$$

Let $\delta_n := (n, n-1, \ldots, 1)$ be the staircase partition with largest part n. There exist integers $d^{(n)}_{\lambda,\mu}$ for $\mu \subseteq \lambda$ such that

$$\prod_{1 \leq i < j \leq n} (1 + x_i + x_j) = 2^{-\binom{n}{2}} \sum_{\mu \subset \delta_{n-1}} 2^{|\mu|} \cdot d^{(n)}_{\delta_{n-1},\mu} \cdot s_\mu(x_1, \ldots, x_n), \text{ and}$$
$$\prod_{1 \leq i \leq j \leq n} (1 + x_i + x_j) = 2^{-\binom{n}{2}} \sum_{\mu \subset \delta_n} 2^{|\mu|} \cdot d^{(n)}_{\delta_n,\mu} \cdot s_\mu(x_1, \ldots, x_n),$$

The integers $d^{(n)}_{\lambda,\mu}$ of Theorem 4.1 are given as follows. Pad λ and μ with 0's so that both sequences $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$ have length n. Assuming $\mu \subseteq \lambda$, the integer $d^{(n)}_{\lambda,\mu}$ is the following determinant of binomial coefficients

$$d^{(n)}_{\lambda,\mu} = \det\left(\binom{\lambda_i + n - i}{\mu_j + n - j} \right)_{1 \leq i,j \leq n}. \tag{4.1}$$

The positivity of this determinant is not obvious. Lascoux [22] gave a geometric proof that $d^{(n)}_{\lambda,\mu} \geq 0$. This determinant was a motivating example for the seminal work of Gessel and Viennot [16]; they gave an interpretation of $d^{(n)}_{\lambda,\mu}$ (and many other such determinants) as counting families of nonintersecting lattice paths.

By the work of Lascoux [22] or Gessel-Viennot [16], the Schur expansions of Theorem 4.1 have nonnegative rational coefficients. In order to deduce that these coefficients are in fact nonnegative integers, observe that the monomial expansions of $\prod_{1 \leq i < j \leq n}(1 + x_i + x_j)$ and $\prod_{1 \leq i \leq j \leq n}(1 + x_i + x_j)$ visibly have positive integer coefficients. Since the transition matrix from the monomial symmetric functions to the Schur functions (the *inverse Kostka matrix*) has integer entries (some of them negative), we conclude that the Schur expansions of Theorem 4.1 have integer coefficients. For example,

$$\prod_{1 \leq i < j \leq 3} (1 + x_i + x_j) = 1 + 2s_{(1)}(X_3) + s_{(2)}(X_3) + 2s_{(1,1)}(X_3) + s_{(2,1)}(X_3). \tag{4.2}$$

A manifestly integral and positive formula for the Schur expansions in Theorem 4.1 may be given as follows. For a partition $\mu \subseteq \delta_{n-1}$, a filling $T : \mu \to \mathbb{Z}_{\geq 0}$ is *reverse flagged* if

- the entries of μ decrease strictly across rows and weakly down columns, and
- the entries in row i of μ lie between 1 and $n - i$.

Let $r_\mu^{(n)}$ be the number of reverse flagged fillings of shape μ. In the case $n = 3$, the collection of reverse flagged fillings of shapes $\mu \subseteq \delta_2 = (2, 1)$ are as follows:

$$\varnothing \quad \boxed{1} \quad \boxed{2} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \, .$$

Compare the shapes in this example to the expansion in (4.2).

Theorem 4.2. *For $n \geq 1$ we have the Schur expansions*

$$\prod_{1 \leq i < j \leq n} (1 + x_i + x_j) = \sum_{\mu \subseteq \delta_{n-1}} r_\mu^{(n)} \cdot s_\mu(X_n), \tag{4.3}$$

$$\prod_{1 \leq i \leq j \leq n} (1 + x_i + x_j) = \sum_{\lambda \subseteq \delta_n} \sum_{\substack{\mu \subseteq \lambda \cap \delta_{n-1} \\ \lambda/\mu \text{ a vertical strip}}} 2^{|\lambda/\mu|} r_\mu^{(n)} \cdot s_\lambda(X_n). \tag{4.4}$$

Recall that the set-theoretic difference λ/μ of Ferrers diagrams is a vertical strip *if no row of λ/μ contains more than one box.*

Proof. We have

$$\prod_{1 \leq i \leq j \leq n} (1 + x_i + x_j) = \left[\sum_{r=0}^{n} 2^r \cdot e_r(X_n) \right] \cdot \prod_{1 \leq i < j \leq n} (1 + x_i + x_j). \tag{4.5}$$

By the *dual Pieri rule*, the Schur expansion of $e_r(X_n) \cdot s_\mu(X_n)$ is obtained by adding a vertical strip of size r to μ in all possible ways, so the second equality follows from the first.

We start with the observation

$$\prod_{1 \leq i < j \leq n} (1 + x_i + x_j) = \prod_{1 \leq i < j \leq n} \frac{x_i(1 + x_i) - x_j(1 + x_j)}{x_i - x_j}. \tag{4.6}$$

Comparing this product with the formula for the Vandermonde determinant and using the antisymmetrizing operators A_n defined in (2.2), we have

$$\prod_{1 \leq i < j \leq n} (1 + x_i + x_j) = A_n \left(\prod_{i=1}^{n} x_i^{n-i} (1 + x_i)^{n-i} \right). \tag{4.7}$$

The antisymmetrizing operator acts linearly, so $\prod_{1 \leq i < j \leq n}(1 + x_i + x_j)$ is the positive sum of terms of the form $c_\alpha A_n(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})$. If $\alpha_i = \alpha_j$ for $i \neq j$, then $A_n(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) = 0$. If the α_i's are all distinct, then there exists a permutation $w \in \mathfrak{S}_n$ and a partition $\mu = (\mu_1, \ldots, \mu_n)$ such that

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = x_1^{\mu_{w(1)} + n - w(1)} \cdots x_n^{\mu_{w(n)} + n - w(n)},$$

hence $A_n(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) = \mathrm{sign}(w) s_\mu(X_n)$. Therefore, in Equation (4.7), the coefficient of $s_\mu(X_n)$ in the Schur expansion of $\prod_{1 \leq i < j \leq n}(1 + x_i + x_j)$ is given by the signed sum

$$\sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) \cdot \left(\begin{array}{c} \text{coefficient of } x_1^{\mu_{w(1)} + n - w(1)} \cdots x_n^{\mu_{w(n)} + n - w(n)} \\ \text{in } \prod_{i=1}^{n} x_i^{n-i} (1 + x_i)^{n-i} \end{array} \right). \tag{4.8}$$

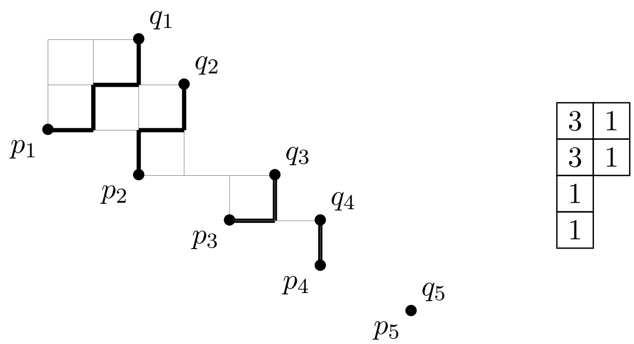
By the Binomial Theorem, the expression in (4.8) is equal to

$$\sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) \cdot \binom{n-1}{\mu_{w(1)} - w(1) + 1} \binom{n-2}{\mu_{w(2)} - w(2) + 2} \cdots \binom{n-n}{\mu_{w(n)} - w(n) + n}. \tag{4.9}$$

In turn, the expression in Equation (4.9) equals the determinant of binomial coefficients

$$\det \left(\binom{n-i}{\mu_j - j + i} \right)_{1 \leq i, j \leq n}. \tag{4.10}$$

We must show that the determinant (4.10) counts reverse flagged fillings of shape $\mu = (\mu_1, \ldots, \mu_n)$ with rows bounded by $(n-1, n-2, \ldots, 0)$. To do this, we use Gessel-Viennot theory [16], see also [33, Sec. 2.7]. For $1 \leq i \leq n$, define lattice points p_i and q_i by $p_i = (2i-2, n-i)$ and $q_i = (n+i-\mu_i-2, n-i+\mu_i)$. The number of paths from p_i to q_j is the binomial coefficient $\binom{n-i}{\mu_j-j+i}$, which is the (i,j)-entry of the determinant (4.10). It follows that the determinant (4.10) counts nonintersecting path families $\mathbb{L} = (L_1, \ldots, L_n)$ such that L_i connects p_i to q_i for all $1 \leq i \leq n$; one such nonintersecting path family is shown below in the $n = 5$ and $\mu = (2, 2, 1, 1, 0)$.



For the final step proving the theorem, we will show there is a bijection from the family of nonintersecting lattice paths with starting points $(p_1, \ldots, p_n)$ and ending points $(q_1, \ldots, q_n)$ to reverse flagged fillings of shape μ with rows bounded by $(n-1, n-2, \ldots, 0)$. Let $\mathbb{L} = (L_1, L_2, \ldots, L_n)$ be such a path family. For $1 \leq i \leq n$, label the edges of the lattice path L_i in order by $1, 2, \ldots, n-i$ starting at q_i and proceeding southwest toward p_i. Observe, the lattice path L_i is completely determined by the subset R_i of edge labels on its vertical edges. Furthermore, $|R_i| = \mu_i$ for each i. Let $F(\mathbb{L})$ be the filling of μ with ith row given by R_i written in decreasing order from left to right. By construction, the entries in the ith row are between 1 and $n-i$. One can check the nonintersecting condition is equivalent to the condition that the columns of $F(\mathbb{L})$ are weakly decreasing. The inverse map is similarly easy to construct from the rows of a reverse flagged filling. Thus, F is the desired bijection, and the expansion in (4.3) holds. An example of this correspondence is shown above. ☐

We note that the number of reverse flagged fillings of μ is effectively calculated by the binomial determinant given in (4.10) and by Lascoux's formula (4.1).

Corollary 4.3. *For $\mu = (\mu_1, \ldots, \mu_n) \subset \delta_{n-1}$,*

$$r_\mu^{(n)} = \det\left(\binom{n-i}{\mu_j - j + i}\right)_{1 \leq i,j \leq n} = \frac{2^{|\mu|}}{2^{\binom{n}{2}}} \cdot \det\left(\binom{2n-2i}{\mu_j + n - j}\right)_{1 \leq i,j \leq n}.$$

We also have the following curious relationship between the coefficients in the Schur expansion of $\prod_{1 \leq i < j \leq n}(1 + x_i + x_j)$ and alternating sign matrices, [23, A005130].

Corollary 4.4. *The sum $\sum_{\mu \subseteq \delta_{n-1}} r_\mu^{(n)} = \prod_{k=0}^{n-1}(3k+1)!/(n+k)!$ which is the number of $n \times n$ alternating sign matrices.*

Proof. From the first determinantal expression for $r_\mu^{(n)}$ in Corollary 4.3, we have

$$\sum_{\mu \subseteq \delta_{n-1}} r_\mu^{(n)} = \sum_{\mu \subseteq \delta_{n-1}} \det\left(\binom{n-i}{\mu_j - j + i}\right)_{1 \leq i,j \leq n}. \tag{4.11}$$

Note, the bottom row of each binomial determinant is all zeros except the (n, n) entry which is 1, hence without changing the sum we can restrict to the determinants of $(n-1) \times (n-1)$ matrices. If we define $r_i = \mu_i + n - i$ for $1 \leq i \leq n-1$, then the expression on the right hand side (up to a minor reindexing) in Equation (4.11) is the expression in [10, Equation 3.1], which enumerates the number of totally symmetric self-complementary plane partitions of $2n$. Such partitions are known to be equinumerous with the set of $n \times n$ alternating sign matrices [1], and the claim follows. □

Remark 4.5. *Reverse flagged fillings show up in Kirillov's work in disguise* [21], *albeit with different motivation. More precisely, Kirillov* [21, Section 5.1] *considers fillings of shape $\mu \subseteq \delta_{n-1}$ that increase weakly along rows and strictly along columns and further satisfy the property that entries in row i belong to the interval $[i, n-1]$. These fillings can be transformed to our reverse flagged fillings by taking transposes and changing each entry j to $n-j$. In view of this transformation, our determinantal formula for reverse flagged fillings is the same as the determinant present in* [21, Theorem 5.6].

Remark 4.6. *The sum of the coefficients in* (4.4) *also give rise to an integer sequence $f(n)$ starting 3, 16, 147, 2304, 61347. This appears to be a new sequence in the literature* [23, A306397]. *We can give this sequence the following combinatorial interpretation. If we denote the number of 1s in a reverse flagged filling T by $m_1(T)$, then this sum of coefficients can be written as*

$$f(n) = \sum_{\lambda \subseteq \delta_n} \sum_{T} 2^{m_1(T)}. \tag{4.12}$$

Here the inner sum runs over all reverse flagged fillings T of shape λ.

Remark 4.7. *Stanley pointed out the following connection between $d^{(n)}_{\lambda,\mu}$ and the number of standard Young tableaux of the skew shape λ/μ, denoted $f^{\lambda/\mu}$,*

$$d^{(n)}_{\lambda,\mu} = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!} \prod_{u \in \lambda/mu} (n + c(u)). \tag{4.13}$$

This is the content of Exercise 78 in [34], *note the numbering is subject to change. Equation* (4.13) *also appears in recent work by Corteel and Kim on lecture hall partitions* [8, Prop. 1.2].

5. A q-analogue of $B_{n,n-1}$ and superspace

In this section, we give representation theoretic models for $B_{n,k}$ in the special case $k = n-1$. Introducing a parameter q, we consider the q-analogue

$$B_{n,n-1}(X_n; q) := \prod_{i=1}^{n} (x_1 + \cdots + x_n + q x_i). \tag{5.1}$$

This specializes to $B_{n,n-1}(x_1, \ldots, x_n; q)$ at $q = -1$. Switching to infinitely many variables, we also consider the symmetric function

$$B_{n,n-1}(X; q) := \sum_{j=0}^{n} q^j \cdot e_j(X) \cdot h_{(1^{n-j})}(X). \tag{5.2}$$

5.1. The specialization $q = 0$.

At $q = 0$, we have the representation theoretic interpretation

$$B_{n,n-1}(X_n; 0) = h_1(X_n)^n = \mathrm{ch}(\overbrace{\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}^{n}), \tag{5.3}$$

where GL_n acts diagonally on the tensor product. The symmetric function $B_{n,n-1}(X; 0) = h_{(1^n)}(X)$ is the Frobenius image $\mathrm{Frob}(\mathbb{C}[\mathfrak{S}_n])$ of the regular representation of $\mathfrak{S}_n$.

5.2. **The specialization** $q=-1$**.** The case $q=-1$ is more interesting. The symmetric function

$$B_{n,n-1}(X;-1) = \sum_{j=0}^{n} (-1)^j \cdot e_j(X) \cdot h_{(1^{n-j})}(X) \tag{5.4}$$

was introduced under the name D_n by Désarménien and Wachs [9] in their study of derangements in the symmetric group. Reiner and Webb [29] described the Schur expansion of $B_{n,n-1}(X;-1)$ in terms of ascents in tableaux. Recall that an *ascent* in a standard Young tableau T with n boxes is an index $1 \le i \le n-1$ such that i appears in a row weakly below $i+1$ in T. Athanasiadis generalized the Reiner-Webb theorem in the context of the $\mathfrak{S}_n$ representation on the homology of the poset of injective words [2].

Theorem 5.1. *(Reiner-Webb [29, Prop. 2.3]) For $n \ge 2$ we have $\omega B_{n,n-1}(X;-1) = \sum_{\lambda \vdash n} a_\lambda s_\lambda$, where a_λ is the number of standard tableaux of shape λ with smallest ascent given by an even number. Here we artificially consider n to be an ascent so every tableau has at least one ascent.*

Gessel and Reutenauer discovered [15, Thm. 3.6] a relationship between $B_{n,n-1}(X_n;-1) = B_{n,n-1}(X_n)$ and free Lie algebras. Specifically, they proved

$$B_{n,n-1}(X_n;-1) = \sum_{\lambda} \mathrm{ch}(\mathrm{Lie}_\lambda(\mathbb{C}^n)). \tag{5.5}$$

The sum ranges over all partitions $\lambda \vdash n$ which have no parts of size 1 and $\mathrm{Lie}_\lambda(\mathbb{C}^n)$ is a GL_n-representation called a *higher Lie module* [28]. Equivalently, if one expands $B_{n,n-1}(X_n;-1)$ into the basis of fundamental quasisymmetric functions, we have

$$B_{n,n-1}(X_n;-1) = \sum_{w \in D_n} F_{D(w)} \tag{5.6}$$

where D_n is the set of derangements in $\mathfrak{S}_n$ and $D(w) = \{i : w(i) > w(i+1)\}$ is the descent set of w. Sundaram [36, Thm. 2.18] introduced a variant $\mathrm{Lie}^{(2)}$ of the Lie modules and obtained another interpretation of $B_{n,n-1}(X_n)$ in terms of them.

5.3. **The specialization** $q=1$**.** At $q=1$, the symmetric function $B_{n,n-1}(X;1)$ has an interpretation involving positroids. A *positroid* of size n is a length n sequence $v_1 v_2 \dots v_n$ consisting of j copies of the letter 0 (for some $0 \le j \le n$) and one copy each of the letters $1, 2, \dots, n-j$. Let P_n denote the set of positroids of size n. For example, we have

$$P_3 = \{123, 213, 132, 231, 312, 321, 012, 021, 102, 201, 120, 210, 001, 010, 100, 000\}.$$

If we use a parameter j to keep track of the number of 0s, we get

$$|P_n| = \sum_{j=0}^{n} \frac{n!}{j!}. \tag{5.7}$$

A more common definition of positroids is permutations in $\mathfrak{S}_n$ with each fixed point colored white or black. More explicitly, the 0's in $v = v_1 \dots v_n \in P_n$ correspond to the white fixed points and the remaining entries of $v_1 \dots v_n$ are order-isomorphic to a unique permutation of the set $\{1 \le i \le n : v_i \ne 0\}$; the fixed points of this smaller permutation are colored black. For example, if $v = 3020041 \in P_7$, then

$$3020041 \leftrightarrow 6234571, \quad \text{with white fixed points } 2,4,5 \text{ and black fixed point } 3.$$

Positroids arise as an indexing set for Postnikov's cellular structure on the totally positive Grassmannian [25].

Let $\mathbb{C}[P_n]$ be the vector space of formal $\mathbb{C}$-linear combinations of elements of P_n. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{C}[P_n]$ as follows. Let $1 \le i \le n-1$, and let $s_i = (i, i+1) \in \mathfrak{S}_n$ be the associated adjacent transposition. If $v = v_1 \dots v_n \in P_n$, then we have $s_i.v := \pm v_1 \dots v_{i+1} v_i \dots v_n$ where the sign is $-$ if $v_i = v_{i+1} = 0$ and $+$ otherwise. As an example, when $n = 4$ we have

$$s_1.(2100) = 1200, \quad s_2.(2100) = 2010, \quad s_3.(2100) = -2100.$$

It can be checked that this rule satisfies the *braid relations*

$$\begin{cases} s_i^2 = 1 & 1 \leq i \leq n-1 \\ s_i s_j = s_j s_i & |i-j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & 1 \leq i \leq n-2 \end{cases} \tag{5.8}$$

and so extends to give an action of $\mathfrak{S}_n$ on $\mathbb{C}[P_n]$.

Proposition 5.2. *We have* $\mathrm{Frob}(\mathbb{C}[P_n]) = B_{n,n-1}(X;1) = \sum_{j=0}^n e_j(X) \cdot h_{(1^{n-j})}(X)$.

Proof. For $0 \leq j \leq n$, let $P_{n,j} \subseteq P_n$ be the family of positroids with j copies of 0. Since the action of $\mathfrak{S}_n$ on P_n does not change the number of 0s, the vector space direct sum $\mathbb{C}[P_n] \cong \bigoplus_{j=0}^n \mathbb{C}[P_{n,j}]$ is stable under the action of $\mathfrak{S}_n$. Since $\mathrm{Frob}(\mathbb{C}[P_n]) = \sum_{j=0}^n \mathrm{Frob}(\mathbb{C}[P_{n,j}])$, it is enough to check that $\mathrm{Frob}(\mathbb{C}[P_{n,j}]) = e_j(X) \cdot h_{(1^{n-j})}(X)$.

By our choice of signs in the action of $\mathfrak{S}_n$ on $\mathbb{C}[P_{n,j}]$ and the definition of induction product, we see that

$$\mathbb{C}[P_{n,j}] \cong \mathrm{sign}_{\mathfrak{S}_j} \circ \mathbb{C}[\mathfrak{S}_{n-j}] \tag{5.9}$$

where $\mathrm{sign}_{\mathfrak{S}_j}$ is the 1-dimensional sign representation $\mathfrak{S}_j$ and $\mathbb{C}[\mathfrak{S}_{n-j}]$ is the regular representation of $\mathfrak{S}_{n-j}$, so that

$$\mathrm{Frob}(\mathbb{C}[P_{n,j}]) = \mathrm{Frob}(\mathrm{sign}_{\mathfrak{S}_j}) \cdot \mathrm{Frob}(\mathbb{C}[\mathfrak{S}_{n-j}]) = e_j(X) \cdot h_{(1^{n-j})}(X), \tag{5.10}$$

as desired. □

We present a graded refinement of $\mathbb{C}[P_n]$ in the next subsection.

5.4. **General q and superspace quotients.** We want to describe a graded $\mathfrak{S}_n$-module whose graded Frobenius image equals $B_{n,n-1}(X;q)$. This module will be a quotient of superspace.

For $n \geq 0$, *superspace* is the associative unital $\mathbb{C}$-algebra with generators $x_1, \dots, x_n, \theta_1, \dots, \theta_n$ subject to the relations

$$x_i x_j = x_j x_i, \quad x_i \theta_j = \theta_j x_i, \quad \theta_i \theta_j = -\theta_j \theta_i \tag{5.11}$$

for all $1 \leq i, j \leq n$. We write $\mathbb{C}[x_1, \dots, x_n, \theta_1, \dots, \theta_n]$ for this algebra, with the understanding that the x-variables commute and the θ-variables anticommute. We can think of this as the ring of polynomial-valued differential forms on $\mathbb{C}^n$. In physics, the x-variables are called *bosonic* and the θ-variables are called *fermionic*. The symmetric group $\mathfrak{S}_n$ acts on superspace diagonally by the rule

$$w.x_i := x_{w(i)}, \quad w.\theta_i := \theta_{w(i)}, \quad w \in \mathfrak{S}_n, \ 1 \leq i \leq n. \tag{5.12}$$

We define the *divergence free* quotient DF_n of superspace by

$$DF_n := \mathbb{C}[x_1, \dots, x_n, \theta_1, \dots, \theta_n]/\langle x_1\theta_1, x_2\theta_2, \dots, x_n\theta_n \rangle. \tag{5.13}$$

Here we think of superspace in terms of differential forms, so that $x_i \theta_i$ is a typical contributor to the divergence of a vector field. The ideal defining DF_n is $\mathfrak{S}_n$-stable and bihomogeneous in the x-variables and the θ-variables, so that DF_n is a bigraded $\mathfrak{S}_n$-module. We use variables t to keep track of x-degree and q to keep track of θ-degree.

What is the bigraded Frobenius image $\mathrm{grFrob}(DF_n; t, q)$? For any subset $J = \{j_1 < \cdots < j_k\} \subseteq [n]$, let $\theta_J := \theta_{j_1} \cdots \theta_{j_k}$ be the corresponding product of θ-variables in increasing order. Also let $\mathbb{C}[X_J]$ be the polynomial ring over $\mathbb{C}$ in the variables $\{x_j : j \in J\}$ with indices in J, so that

$\mathbb{C}[X_{[n]-J}]$ is the polynomial ring with variables whose indices do *not* lie in J. We have a vector space direct sum decomposition

$$DF_n = \bigoplus_{J \subseteq [n]} \mathbb{C}[X_{[n]-J}] \cdot \theta_J. \tag{5.14}$$

Let $DF_J := \mathbb{C}[X_{[n]-J}] \cdot \theta_J$ be the summand in (5.14) corresponding to J. The spaces DF_J are not closed under the action of $\mathfrak{S}_n$ unless $J = \varnothing$ or $J = [n]$. To fix this, for $0 \leq j \leq n$ we set $DF_{n,j} := \bigoplus_{|J|=j} DF_J$. By (5.14) we have $DF_n = \bigoplus_{j=0}^{n} DF_{n,j}$. Recall the plethystic formula for the graded Frobenius image of the polynomial ring:

$$\mathrm{grFrob}(\mathbb{C}[X_n];t) = h_n\left[\frac{X}{1-t}\right]. \tag{5.15}$$

By the definition of induction product and Equation (5.15) we have

$$\mathrm{grFrob}(DF_{n,j};q,t) = q^j \cdot e_j(X) \cdot h_{n-j}\left[\frac{X}{1-t}\right], \tag{5.16}$$

$$\mathrm{grFrob}(DF_n;q,t) = \sum_{j=0}^{n} q^j \cdot e_j(X) \cdot h_{n-j}\left[\frac{X}{1-t}\right]. \tag{5.17}$$

Let $I_n = \langle e_1(X_n), e_2(X_n), \ldots, e_n(X_n) \rangle \subseteq DF_n$ be the ideal generated by the n elementary symmetric polynomials in the x-variables. Equivalently, we can think of I_n as the ideal generated by the vector space $\mathbb{C}[X_n]_+^{\mathfrak{S}_n}$ of symmetric polynomials with vanishing constant term within the divergence free quotient of superspace. Let $R_n := DF_n/I_n$ be the corresponding bigraded quotient $\mathfrak{S}_n$-module.

Theorem 5.3. *The bigraded Frobenius image of R_n is*

$$\mathrm{grFrob}(R_n;q,t) = \sum_{j=0}^{n} q^j \cdot e_j(X) \cdot \left[\sum_{T \in \mathrm{SYT}(n-j)} t^{\mathrm{maj}(T)} \cdot s_{\mathrm{shape}(T)}(X)\right], \tag{5.18}$$

where the sum is over all standard Young tableaux T with $n-j$ boxes. Consequently, we have

$$\mathrm{grFrob}(R_n;q,1) = \sum_{j=0}^{n} q^j \cdot e_j(X) \cdot h_{(1^{n-j})}(X) = B_{n,n-1}(X;q). \tag{5.19}$$

In the case $n = 3$, the standard tableaux with ≤ 3 boxes are as follows:

$$\varnothing \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}.$$

From left to right, their major indices are $0,0,0,1,0,2,1,3$. By Equation (5.18),

$$\mathrm{grFrob}(R_n;q,t) = q^3 e_3 + q^2 e_2 s_1 + q e_1 s_2 + q t e_1 s_{11} + s_3 + t s_{21} + t^2 s_{21} + t^3 s_{111}.$$

Proof. We can relate $DF_\varnothing = \mathbb{C}[X_n]$ to the other DF_J using maps. Specifically, define

$$\varphi_J : DF_\varnothing \to DF_J \tag{5.20}$$

by the rule $\varphi_J(f(X_n)) = f(X_n)\theta_J$. Each φ_J is a map of $\mathbb{C}[X_n]$-modules.

Let $I_n' \subseteq \mathbb{C}[X_n]$ be the classical *invariant ideal* $I_n' := \langle e_1(X_n), e_2(X_n), \ldots, e_n(X_n) \rangle$ which has the same generating set as I_n, but is generated in the subring $\mathbb{C}[X_n] \subseteq DF_n$. The ideals I_n and I_n' may be related as follows: for any subset $J \subseteq [n]$,

$$I_n \cap DF_J = \varphi_J(I_n') \tag{5.21}$$

and furthermore there hold the graded vector space decompositions

$$I_n = \bigoplus_{J \subseteq [n]} (I_n \cap DF_J) = \bigoplus_{J \subseteq [n]} \varphi_J(I'_n). \tag{5.22}$$

Taking quotients, this gives

$$R_n = \bigoplus_{J \subseteq [n]} DF_J/(I_n \cap DF_J) = \bigoplus_{J \subseteq [n]} DF_J/\varphi_J(I'_n). \tag{5.23}$$

What does the quotient $DF_J/\varphi_J(I'_n)$ look like? Since $x_i\theta_i = 0$ in DF_n for all i, for any $1 \leq k \leq n$ we have

$$\varphi_J(e_k(X_n)) = e_k(X_{[n]-J}) \cdot \theta_J, \tag{5.24}$$

where $e_k(X_{[n]-J})$ is the degree k elementary symmetric polynomial in the variable set indexed by $[n]-J$; observe that $e_k(X_{[n]-J}) = 0$ whenever $k > n - |J|$. Since φ_J is a map of $\mathbb{C}[X_n]$-modules, the polynomials $e_k(X_{[n]-J})$ for $0 < k \leq n - |J|$ form a generating set for the $\mathbb{C}[X_n]$-module $\varphi_J(I'_n)$. Consequently, the map

$$\mathbb{C}[X_{[n]-J}]/\langle e_k(X_{[n]-J}) \,:\, 0 < k \leq [n] - J \rangle \xrightarrow{\cdot \theta_J} DF_J/\varphi_J(I'_n) \tag{5.25}$$

given by multiplication by θ_J is a $\mathbb{C}$-linear isomorphism which maps a homogeneous polynomial of bidegree (q,t) to $(q+j,t)$.

The reasoning of the last paragraph shows that R_n admits a direct sum decomposition as a bigraded vector space:

$$R_n \cong \bigoplus_{J \subseteq [n]} \mathbb{C}[X_{[n]-J}]/\langle e_k(X_{[n]-J}) \,:\, 0 < k \leq n - |J| \rangle \otimes \mathbb{C}\{\theta_J\}, \tag{5.26}$$

where $\mathbb{C}\{\theta_J\}$ is the 1-dimensional $\mathbb{C}$-vector space spanned by θ_J. This may also be expressed with induction product as a bigraded $\mathfrak{S}_n$-module isomorphism

$$R_n \cong \bigoplus_{j=0}^{n} \mathbb{C}[X_{n-j}]/I'_{n-j} \circ \mathbb{C}\{\theta_1\theta_2 \cdots \theta_j\}. \tag{5.27}$$

Since $\mathbb{C}\{\theta_1\theta_2 \dots \theta_j\}$ carries the sign representation of $\mathfrak{S}_j$ in q-degree j and $\mathbb{C}[X_{n-j}]/I'_{n-j}$ is the classical coinvariant ring corresponding to $\mathfrak{S}_{n-j}$, the claimed Frobenius image now follows from Theorem 2.1. ☐

Remark 5.4. *For $n > 0$, Reiner and Webb* [29] *consider a chain complex*

$$C_\bullet = (\cdots \to C_2 \to C_1 \to C_0 \to 0)$$

whose degree j component C_j has basis given by length j words $a_1 \dots a_j$ over the alphabet $[n]$ with no repeated letters. Up to a sign twist, the polynomial

$$\sum_{j=0}^{n} q^j \cdot \mathrm{Frob}(C_j) \tag{5.28}$$

coming from the graded action of $\mathfrak{S}_n$ on $C_\bullet$ (without taking homology) equals $B_{n,n-1}(X;q)$. At $q = 1$ (again up to a sign twist) we get the action of $\mathfrak{S}_n$ on the space $\mathbb{C}[P_n]$ spanned by positroids of Proposition 5.2.

Rhoades and Wilson [30] *have an alternative representation-theoretic interpretation of the q-analog $B_{n,n-1}(X;q)$ as well as the q,t-generalization of Theorem 5.3 which is defined using an extension of the Vandermonde determinant to superspace.*

The key fact in the proof of Theorem 5.3 was that a $\mathbb{C}[X_n]$-module generating set for $I_n \cap DF_J$ can be obtained by applying the map φ_J to the generators of the $\mathbb{C}[X_n]$-module $I_n \cap DF_\varnothing$, or equivalently the generators of the ideal $I'_n \subseteq \mathbb{C}[X_n]$. Since the images of these generators under φ_J have a nice form, it was possible to describe the J-component $DF_n/\varphi_J(I'_n) = DF_n/(I_n \cap DF_J)$ of R_n.

The program of the above paragraph can be carried out for a wider class of ideals. For $r \leq k \leq n$, consider the ideal $I_{n,k,r} \subseteq DF_n$ with generators

$$I_{n,k,r} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n(X_n), e_{n-1}(X_n), \ldots, e_{n-r+1}(X_n) \rangle. \tag{5.29}$$

The ideal

$$I'_{n,k,r} := I_{n,k,r} \cap DF_\varnothing = I_{n,k,r} \cap \mathbb{C}[X_n] \tag{5.30}$$

was defined by Haglund, Rhoades, and Shimozono and gives a variant of the coinvariant ring whose properties are governed by ordered set partitions [19]. Pawlowski and Rhoades proved that the quotient of $\mathbb{C}[X_n]$ by $I'_{n,k,r}$ and presents the cohomology of a certain variety of line configurations [24] (denoted $X_{n,k,r}$ therein).

Let $R_{n,k,r} := DF_n/I_{n,k,r}$ be the quotient of DF_n by $I_{n,k,r}$. The same reasoning as in the proof of Theorem 5.3 gives

$$R_{n,k,r} \cong \bigoplus_{j=0}^{n} \mathbb{C}[X_{n-j}]/I'_{n-j,k,r-j} \circ \mathbb{C}\{\theta_1\theta_2\cdots\theta_j\}, \tag{5.31}$$

so that

$$\mathrm{grFrob}(R_{n,k,r}; q, t) = \sum_{j=0}^{n} q^j \cdot e_j(X) \cdot \mathrm{grFrob}(\mathbb{C}[X_{n-j}]/I'_{n-j,k,r-j}; t). \tag{5.32}$$

The Schur expansion of the symmetric function (5.32) follows from material in [19, Sec. 6]. Each term on the right-hand-side of Equation (5.32) has the form $\mathrm{grFrob}(\mathbb{C}[X_n]/I'_{n,k,r}; t)$ for some $r \leq k \leq n$. Combining [19, Lem. 6.10] and [19, Cor. 6.13] we have

$$\mathrm{grFrob}(\mathbb{C}[X_n]/I'_{n,k,r}; t) = \sum_{m=0}^{k-r} t^{m\cdot(n-k+m)} \begin{bmatrix} k-r \\ m \end{bmatrix}_t \left(\sum_{T \in \mathrm{SYT}(n)} t^{\mathrm{maj}(T)} \begin{bmatrix} n-\mathrm{des}(T)-1 \\ n-k+m \end{bmatrix}_t s_{\mathrm{shape}(T)}(X) \right). \tag{5.33}$$

Here we adopt the t-binomial notation

$$\begin{bmatrix} n \\ k \end{bmatrix}_t := \frac{[n]_t!}{[k]!_t[n-k]!_t}, \quad [n]!_t := [n]_t[n-1]_t \cdots [1]_t, \quad [n]_t := 1 + t + \cdots + t^{n-1}. \tag{5.34}$$

6. Conclusion

As an extension of Problem 3.10, one could ask for a module whose Weyl character is given by the expression in Pragacz's Theorem 3.7. For simplicity, let us consider the case of one rank n vector bundle $\mathcal{E}$ with Chern roots $x_1, \ldots, x_n$ and the Chern plethysm $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$ for two partitions λ and μ. If W is a GL_n-module with Weyl character $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$, then

$$\dim W = s_\lambda(\mathbb{S}^\mu(\mathcal{E}))\mid_{x_1=\cdots=x_n=1} = |\mu|^{|\lambda|} \cdot |\mathrm{SSYT}(\lambda, \leq |\mathrm{SSYT}(\mu, \leq n)|)|, \tag{6.1}$$

where the second equality uses the fact that $\mathbb{S}^\mu(\mathcal{E})$ has Chern roots $\sum_{\square\in\mu} x_{T(\square)}$ where T ranges over all elements of $\mathrm{SSYT}(\mu, \leq n)$.

The quantity $|\mathrm{SSYT}(\lambda, \leq |\mathrm{SSYT}(\mu, \leq n)|)|$ has a natural representation theoretic interpretation via Schur functor composition:

$$\dim \mathbb{S}^\lambda(\mathbb{S}^\mu(\mathbb{C}^n)) = |\mathrm{SSYT}(\lambda, \leq |\mathrm{SSYT}(\mu, \leq n)|)|. \tag{6.2}$$

This suggests the following problem.

Problem 6.1. *Let W be the vector space* $\mathrm{Hom}_{\mathbb{C}}(\mathbb{S}^\lambda(\mathbb{S}^\mu(\mathbb{C}^n)), (\mathbb{C}^{|\mu|})^{\otimes|\lambda|})$. *Find an action of GL_n on W whose Weyl character equals $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$.*

The natural GL_n-action on W coming from acting on $\mathbb{C}^n$ does not solve Problem 6.1. Indeed, this is a polynomial representation of GL_n of degree $|\lambda| \cdot |\mu|$ whereas the polynomial $s_\lambda(\mathbb{S}^\mu(\mathcal{E}))$ has degree $|\lambda|$. We remark that the Weyl character of the GL_n-action on $\mathbb{S}^\lambda(\mathbb{S}^\mu(\mathbb{C}^n))$ coming from the action of GL_n on $\mathbb{C}^n$ is the classical plethysm $s_\lambda[s_\mu]$. Problem 6.1 asks for the corresponding representation theoretic operation for Chern plethysm.

We close with some connections between Boolean product polynomials and the theory of maximal unbalanced collections. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ and for $S \subseteq [n]$, let $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i$ be the sum of the coordinate vectors in S. A collection of subsets $\mathcal{C} \subseteq 2^{[n]}$ is called *balanced* if the convex hull of the vectors $\mathbf{e}_S$ for $S \in \mathcal{C}$ meets the main diagonal $\{(t, t, \ldots, t) : 0 \leq t \leq 1\}$ in $[0, 1]^n$. Otherwise, the collection $\mathcal{C}$ is *unbalanced.*

Balanced collections were defined by Shapley [31] in his study of n-person cooperative games. In the containment partial order on $2^{[n]}$, balanced collections form an order filter and unbalanced collections form an order ideal, so we can consider *minimal balanced* and *maximal unbalanced* collections. Minimal balanced collections were considered by Shapley [31] and maximal unbalanced collections arose independently in the work of Billera-Moore-Moraites-Wang-Williams [4] and Björner [5]. In particular, Billera et. al. gave a bijection between maximal unbalanced collections and the regions of the resonance arrangement [4]. Thus, counting maximal unbalanced collections is equivalent to counting the chambers in the resonance arrangement defined by the polynomial $B_n(X_n)$.

One way to count the chambers of the resonance arrangement would be to find the characteristic polynomial of the matroid $M_n = \{\mathbf{e}_S : \varnothing \neq S \subseteq [n]\}$. To understand the matroid M_n, one would need to know whether the determinant $\det A$ of any $0, 1$-matrix A of size $n \times n$ is zero or not. Determinants of $0, 1$-matrices arise [37] in *Hadamard's maximal determinant problem*, which asks whether there exists an $n \times n$ $0, 1$-matrix A such that $\det A = (n+1)^{(n+1)/2}/2^n$ (the inequality $\leq$ is known to hold for any matrix A). The study of the matroid M_n could shed light on the Hadamard problem.

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