

GESSEL POLYNOMIALS, ROOKS, AND EXTENDED LINIAL ARRANGEMENTS

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ABSTRACT. We study a family of polynomials associated with ascent-descent statistics on labeled rooted plane k -ary trees introduced by Gessel, from a rook-theoretic perspective. We generalize the excedance statistic on permutations to maximal nonattacking rook placements on certain rectangular boards by decomposing them into staircase boards. We then relate the number of maximal nonattacking rook placements on certain skew boards to the number of regions in extended Linial arrangements by establishing a relation between the factorial polynomial of those boards to the characteristic polynomial of extended Linial arrangements. Furthermore, we give a combinatorial interpretation of the number of bounded regions in extended Linial arrangements in terms of certain labeled rooted plane k -ary trees. Finally, using the work of Goldman-Joichi-White, we identify graphs whose chromatic polynomials equal the characteristic polynomials of extended Linial arrangements up to a straightforward normalization.

1. INTRODUCTION

This article concerns a family of multivariate polynomials associated with ascents and descents on labeled rooted plane k -ary trees that was introduced by Gessel [13] and studied further in [10, 21]. Gessel observed that various evaluations of these polynomials counted the number of regions in well-known deformations of Coxeter arrangements and asked for a bijective/combinatorial explanation [14]. While Bernardi [7] recently answered Gessel's question in great generality, we describe an alternate perspective on the same which we believe is meaningful. We study the aforementioned family of polynomials from a rook-theoretic perspective, drawing inspiration from recent work of Lewis-Morales [24] studying diagrams of permutations as well as the work of Leven-Rhoades-Wilson [23] relating rook placements to Shi-Ish arrangements.

To give a flavor of our main results, we establish some notation. Let $n \geq 1$. A local binary search tree on n nodes is a labeled rooted plane binary tree on n nodes with distinct labels drawn from $[n] = \{1, \dots, n\}$ such that the label of the left child (respectively right child) is always smaller (respectively greater) than the parent. The Linial arrangement Lin_{n-1} is the real hyperplane arrangement defined by hyperplanes $x_i - x_j = 1$ for $1 \leq i < j \leq n$ considered in the ambient vector space V_{n-1} consisting of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_1 + \dots + x_n = 0$. Let B denote the skew Ferrers board of shape $(2n - 3, \dots, n - 1)/(n - 1, \dots, 1)$. A special case of one of our main theorems is the following result.

Theorem 1.1. *The number of bounded regions in Lin_{n-1} equals the following quantities.*

- (1) *The number of local binary search trees on n nodes such that the nodes labeled 1 and n have both their children being unlabeled leaves (see Figure 9 for the case $n = 4$).*
- (2) *The number of placements of $n - 1$ nonattacking rooks on the board B .*
- (3) *The number of sequences (x_1, \dots, x_{n-1}) of positive integers satisfying $1 \leq x_i \leq n - 2$ for $1 \leq i \leq n - 1$, and $x_i + i \neq x_j + j$ for $1 \leq i \neq j \leq n - 1$.*

A few remarks about Theorem 1.1 are in order. The question of finding a combinatorial meaning of the number of bounded regions of the Linial arrangement in the setting of labeled trees has been raised in [2] (cf. discussion after Theorem 4.2 therein). To the best of the author's knowledge, there have not been

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any results in this direction. Secondly, the relation between local binary search trees and regions in the Linial arrangement is no surprise. Indeed, Postnikov [26] proved that the number of regions in the Linial arrangement equals the number of local binary search trees.[†] As Theorem 1.1 shows, the number of bounded regions in the Linial arrangement is counted by a nicely described subset of the set of local binary search trees. Starting from $n = 1$, the first few terms of this sequence are: 1, 0, 1, 4, 26, 212, 2108, 24720.

Our second main result relates the factorial polynomial of the board B from earlier to the characteristic polynomial of Lin_{n-1} .

Theorem 1.2. *Let $R(t, B)$ denote the factorial polynomial of B and $\chi_{\text{Lin}_{n-1}}(t)$ denote the characteristic polynomial of Lin_{n-1} . Then*

$$R(t, B) = (-1)^{n-1} \chi_{\text{Lin}_{n-1}}(1-t).$$

We work with the convention that the factorial polynomial of the empty board, which corresponds to the case $n = 1$ in Theorem 1.2, is 1.

The outline of this article is as follows: We begin by defining notions related to hyperplane arrangements in Section 2 and follow it up by defining rook-theoretic notions in Section 3. We introduce certain rectangular boards and define excedance and subcedance statistics associated with maximal nonattacking rook placements on these boards. In Section 4, we introduce labeled rooted plane k -ary trees amongst other notions. We then define ascent and descent statistics on these trees and proceed to give our weight-preserving bijection Ψ that takes maximal nonattacking rook placements on $(n-1) \times kn$ boards to labeled rooted plane k -ary trees with distinct labels drawn from $[n]$ so that the subcedance-excedance statistics associated with the rook placement get mapped to the ascent-descent statistics associated with the corresponding tree. This bijection allows us to define a new family of multivariate polynomials $G_{n,k}$ that we call Gessel polynomials. In Section 5, we employ our bijection to relate rook placements on certain skew Ferrers boards to the number of regions in extended Linial arrangements. In Section 5.2 we identify graphs whose chromatic polynomials are related to characteristic polynomials of extended Linial arrangements. We conclude with a brief discussion on future avenues in Section 6.

2. HYPERPLANE ARRANGEMENTS

We begin by defining some basic notions concerning hyperplane arrangements. Our exposition is brief; the reader is referred to [30, 35] for more details. To a large extent, we adhere to the terminology in [28].

A *hyperplane arrangement* is a finite collection of affine hyperplanes in a vector space. Given a finite-dimensional vector space V over \mathbb{R} , let \mathcal{A} be a hyperplane arrangement therein. We denote the number of *regions* of \mathcal{A} , that is, the number of connected components of $V - \cup_{H \in \mathcal{A}} H$, by $r(\mathcal{A})$. Furthermore, the number of *relatively bounded regions* is denoted by $b(\mathcal{A})$. The *intersection poset* $L_{\mathcal{A}}$ associated to \mathcal{A} connects the study of hyperplane arrangements to combinatorics. It consists of all nonempty intersections $H_{i_1} \cap \cdots \cap H_{i_k}$ of hyperplanes in \mathcal{A} ordered by reverse inclusion. The unique minimum element $\hat{0}$ in $L_{\mathcal{A}}$ is the ambient vector space V . We define the *characteristic polynomial* of \mathcal{A} , denoted by $\chi_{\mathcal{A}}(q)$, as

$$\chi_{\mathcal{A}}(q) = \sum_{z \in L_{\mathcal{A}}} \mu(\hat{0}, z) q^{\dim(z)},$$

where μ denotes the Möbius function of $L_{\mathcal{A}}$. The following seminal result of Zaslavsky [37] connects the enumeration of regions in a real hyperplane arrangement to evaluations of its characteristic polynomial.

Theorem 2.1. *For a hyperplane arrangement \mathcal{A} in \mathbb{R}^n , we have $r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$ and $b(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(1)$.*

[†]In fact, Postnikov works with intransitive trees or alternating trees, but establishes a bijection between them and local binary search trees [26, Section 4.1].

We turn our focus to certain hyperplane arrangements that are our primary object of study. Given a positive integer n , let V_{n-1} be the subspace in \mathbb{R}^n comprising vectors (x_1, \dots, x_n) such that $\sum_{i=1}^n x_i = 0$. All hyperplane arrangements we consider lie in V_{n-1} and, in fact, are special instances of so-called *truncated affine arrangements* [28]. Furthermore, they are *essential* and henceforth we will simply say bounded regions instead of relatively bounded regions.

Fix nonnegative integers a and b such that $a + b \geq 2$, and consider the *truncated affine arrangement* $\mathcal{A}_{n-1}^{a,b}$ given by

$$x_i - x_j = -a + 1, -a + 2, \dots, b - 1, \quad 1 \leq i < j \leq n.$$

We denote its characteristic polynomial by $\chi_{n-1}^{a,b}(q)$. Specific instances of truncated affine arrangements have been a focus of intense study in recent years and have resulted in a lot of interesting combinatorics. We describe these arrangements next. Fix a positive integer $a \geq 1$. The arrangement $\mathcal{A}_{n-1}^{a,a}$ is called the *extended Catalan arrangement* and denoted by $\mathcal{C}_{n-1,a}$. The case $a = 1$ corresponds to the braid arrangement, whereas the case $a = 2$ corresponds to the classical Catalan arrangement. The arrangement $\mathcal{A}_{n-1}^{a,a+1}$ is referred to as the *extended Shi arrangement* and denoted by $\mathcal{S}_{n-1,a}$. The case $a = 1$ corresponds the extremely well-studied Shi arrangement [6, 18, 28, 32, 33, 36]. Finally the arrangement $\mathcal{A}_{n-1}^{a-1,a+1}$ is referred to as the *extended Linial arrangement* and denoted by $\text{Lin}_{n-1,a}$. The case $a = 1$ corresponds to the Linial arrangement mentioned in the introduction. From now on, $\text{Lin}_{n-1} := \text{Lin}_{n-1,1}$. Before proceeding to our next section, we remark that various aspects of the truncated affine arrangements have been studied in great depth in [3, 4, 5, 28].

3. ROOK PLACEMENTS AND EXCEDANCE-SUBCEDANCE STATISTICS

A *board* B is a finite subset of $\mathbb{P} \times \mathbb{P}$ (thought of in Cartesian coordinates) where \mathbb{P} denotes the set of positive integers. We label the rows and columns of a board with positive integers. By row i we mean the i -th row from the bottom and by column j we mean the j -th column from the left. We are particularly interested in *skew Ferrers boards*. These are boards corresponding to Young diagrams of shape λ/μ for partitions λ and μ satisfying $\mu \subseteq \lambda$.

A *nonattacking rook placement* on B is a placement where there is at most one rook in any row or column. It is said to be *maximal* if there is exactly one rook in every row and at most one in any column. Given a nonnegative integer k , let $r_k(B)$ denote the number of nonattacking rook placements of k rooks on B . Clearly, if k exceeds the number of rows in B , then $r_k(B) = 0$. Also, $r_0(B) = 1$. If m is greater than or equal to the number of rows in B , then we denote by $R_m(x, B)$ the *m -factorial polynomial* of B [15] defined by

$$R_m(x, B) = \sum_{k \geq 0} r_k(B)(x)_{m-k}.$$

Here $(x)_j$ is the falling factorial defined to be $x(x-1)\cdots(x-j+1)$ if $j \geq 1$ and 1 otherwise. We usually set m equal to the number of rows in the board B . In this case we refer to the m -factorial polynomial of B as the *factorial polynomial* of B , and denote it by $R(x, B)$. Finally, if B is the empty board, then $R(x, B) := 1$. For the board B in Figure 1, one may verify that $R(x, B) = (x)_3 + 9(x)_2 + 22(x)_1 + 14$. It is worth remarking

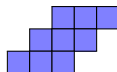
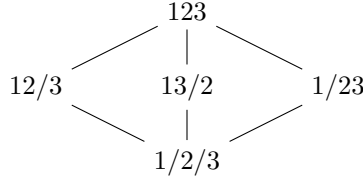


FIGURE 1. A skew Ferrers board of shape $(5, 4, 3)/(2, 1)$

that the coefficients 1, 9, 22, and 14 appear in the analysis of the ‘three-ply’ staircase in Riordan’s work [31]. The board B in Figure 1 is a special instance of boards considered by Riordan.

FIGURE 2. Hasse diagram of Π_3 .

Recall that Theorem 1.2 relates the characteristic polynomial of Lin_{n-1} to the factorial polynomial of a certain skew Ferrers board. Since the characteristic polynomial of a hyperplane arrangement is defined using the Möbius function of its intersection lattice, it would be helpful to have a similar description for factorial polynomials. This is present in the beautiful work of Goldman-Joichi-White [16]. Let Π_m denote the lattice of partitions of $[m]$, and let $\hat{0}$ denote the unique minimum in Π_m . Consider a board B with m rows. Given a subset $S \subseteq [m]$, we define a statistic $v_B(S)$ as follows:

$$v_B(S) = |\{j \mid \forall i \in S, (i, j) \in B\}|.$$

Theorem 3.1 (Goldman-Joichi-White). *Let B be a board with m rows. Then*

$$R(x, B) = \sum_{\sigma \in \Pi_m} \mu(\hat{0}, \sigma) \prod_{A \in \sigma} (x + v_B(A))$$

where μ denotes the Möbius function of Π_m , and by $A \in \sigma$ we mean that A is a block in σ .

Figure 2 depicts the Hasse diagram of Π_3 where the commas and braces in writing the blocks are omitted. By Theorem 3.1, for the board B in Figure 1, we obtain $R(x, B) = (x+3)^3 - (x+2)(x+3) - (x+1)(x+3) - (x+2)(x+3) + 2(x+1) = x^3 + 6x^2 + 15x + 14$. In particular, note that $R(1, B) = 36$ and $R(-1, B) = 4$, which coincide with the number of regions and the number of bounded regions in Lin_3 respectively.

3.1. Excedance-subcedance statistics and rectangular boards. To define certain statistics on nonattacking rook placements on rectangular boards, we interpret them in terms of functions as described next. Given positive integers n, k , let $\mathcal{F}_{n,k}$ be defined as

$$\mathcal{F}_{n,k} = \{f : [n-1] \hookrightarrow [kn]\}.$$

Thus, $\mathcal{F}_{n,k}$ consists of injective functions from the first $n-1$ positive integers to the first kn positive integers. We work with the convention that $[n-1]$ is the empty set \emptyset when $n=1$. We identify an element $f \in \mathcal{F}_{n,k}$ with the maximal nonattacking rook placement on an $(n-1) \times kn$ rectangular board by placing a rook in position $(i, f(i))$ for $1 \leq i \leq n-1$. Clearly, statements involving rook placements on sub-boards of the $(n-1) \times kn$ board can be phrased in terms of functions in $\mathcal{F}_{n,k}$ satisfying appropriate constraints. Note that there is a unique injective function from the empty set \emptyset (corresponding to the case $n=1$) into $[k]$ for a positive integer k . This corresponds to the unique placement of 0 rooks on the empty board. To avoid unnecessary complications, we assume henceforth that $n \geq 2$.

Closely associated to $\mathcal{F}_{n,k}$ is the set $\tilde{\mathcal{F}}_{n,k}$ defined as

$$\tilde{\mathcal{F}}_{n,k} = \{f : [n-1] \hookrightarrow [n] \times [k]\}.$$

Let $\Phi : \mathcal{F}_{n,k} \rightarrow \tilde{\mathcal{F}}_{n,k}$ be the map that sends $f \in \mathcal{F}_{n,k}$ to $\Phi(f)$ defined by

$$\Phi(f)(i) = \left(f(i) - n \left\lfloor \frac{f(i) - 1}{n} \right\rfloor, \left\lfloor \frac{f(i)}{n} \right\rfloor \right) \text{ for all } 1 \leq i \leq n-1.$$

It is easy to see that Φ is a bijection between $\mathcal{F}_{n,k}$ and $\tilde{\mathcal{F}}_{n,k}$. Figure 3 depicts an $f \in \mathcal{F}_{6,2}$ as a maximal nonattacking rook placement on the left. The rook placement on the right corresponds to $g := \Phi(f) \in \tilde{\mathcal{F}}_{6,2}$. Note that g can be interpreted as a maximal rook placement of ‘colored’ rooks, and it is ‘nonattacking’ in

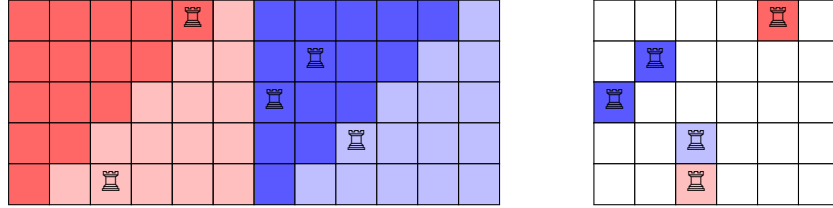


FIGURE 3. An placement in $\mathcal{F}_{6,2}$ on the left and the associated placement in $\tilde{\mathcal{F}}_{6,2}$ on the right.

the sense that if two rooks occupy the same column, then they possess distinct colors.

Given $g \in \tilde{\mathcal{F}}_{n,k}$, we say that $1 \leq i \leq n - 1$ is a j -*excedance* (respectively j -*subcedance*) if $g(i)$ equals (a, j) for some positive integer a satisfying $a > i$ (respectively $a \leq i$). This given, we associate two weak compositions with $g \in \tilde{\mathcal{F}}_{n,k}$,

$$\begin{aligned} \text{exc}(g) &= (\alpha_1, \dots, \alpha_k) \\ \text{sub}(g) &= (\beta_1, \dots, \beta_k), \end{aligned}$$

where α_j (respectively β_j) counts the number of j -excedances (respectively j -subcedances) in g . Note that $|\text{exc}(g)| + |\text{sub}(g)| = n - 1$. Let $\mathbf{u} = \{u_1, \dots, u_k\}$ and $\mathbf{v} = \{v_1, \dots, v_k\}$ be two sets of commuting indeterminates, which further commute with each other. Associate the weight $\mathbf{u}^{\text{exc}(g)} \mathbf{v}^{\text{sub}(g)} := u_1^{\alpha_1} \dots u_k^{\alpha_k} v_1^{\beta_1} \dots v_k^{\beta_k}$ with g . For the g in Figure 3, we have $\text{exc}(g) = (1, 1)$ and $\text{sub}(g) = (1, 2)$, and the weight of g is $u_1 u_2 v_1 v_2^2$. Finally, we associate a polynomial $E_{n,k}(\mathbf{u}, \mathbf{v})$ in the variables $u_1, \dots, u_k, v_1, \dots, v_k$ with $\tilde{\mathcal{F}}_{n,k}$ as follows:

$$E_{n,k}(\mathbf{u}, \mathbf{v}) = \sum_{g \in \tilde{\mathcal{F}}_{n,k}} \mathbf{u}^{\text{exc}(g)} \mathbf{v}^{\text{sub}(g)}.$$

Note that $E_{n,1}$ tracks classical excedance-subcedance statistics over permutations of n letters.

3.2. Catalan, Shi, and Linial boards. We introduce certain Ferrers boards whose factorial polynomials will be interpreted as characteristic polynomials of certain truncated affine arrangements. Let t be a nonnegative integer. The t -*Catalan board*, denoted by $C_{t,n}$, is defined to be the $(n - 1) \times (n - 2 + t)$ board. The t -*Shi board*, denoted by $S_{t,n}$, is defined to be the Ferrers board of shape $(2n - 3 + t, \dots, n - 1 + t)$. Finally, the t -*Linial board*, denoted by $L_{t,n}$, is defined to be the skew Ferrers board of shape $(2n - 3 + t, \dots, n - 1 + t) / (n - 1, \dots, 1)$. Figure 4 depicts the boards $C_{t,n}$, $S_{t,n}$ and $L_{t,n}$ for $n = 3$ and $t = 2$.

An easy computation establishes that

$$\begin{aligned} r_{n-1}(C_{t,n}) &= \prod_{j=1}^{n-1} (t + n - 1 - j) = \sum_{j=0}^{n-1} r_j(C_{0,n})(t)_{n-1-j} = R(t, C_{0,n}), \\ r_{n-1}(S_{t,n}) &= (t + n - 1)^{n-1} = \sum_{j=0}^{n-1} r_j(S_{0,n})(t)_{n-1-j} = R(t, S_{0,n}). \end{aligned}$$

On the other hand, from [18, Theorem 2.4] (also from [28, Corollary 9.9.1] and [2, Theorem 3.3]), it follows that $\chi_{n-1}^{1,2}(q) = (q - n)^{n-1}$. Similarly, from [28, Theorem 9.8], it follows that $\chi_{n-1}^{1,1}(q) = (q + 1 - n)(q +$

$2 - n) \cdots (q + n - 1 - n)$. We therefore infer the following equalities relating characteristic polynomials of extended Catalan and Shi arrangements to factorial polynomials of Catalan and Shi boards respectively.

$$(3.1) \quad \begin{aligned} R(t, C_{0,n}) &= (-1)^{n-1} \chi_{n-1}^{1,1}(1-t), \\ R(t, S_{0,n}) &= (-1)^{n-1} \chi_{n-1}^{1,2}(1-t). \end{aligned}$$

The reader is invited to compare Equation (3.1) with the statement of Theorem 1.2. Note also that Theorem 3.1 allows us to write characteristic polynomials of extended Catalan and Shi arrangements as sums over elements of Π_{n-1} as follows.

$$(3.2) \quad \begin{aligned} R(t, C_{0,n}) &= \sum_{\sigma \in \Pi_{n-1}} \mu(\hat{0}, \sigma) \prod_{A \in \sigma} (t + n - 2), \\ R(t, S_{0,n}) &= \sum_{\sigma \in \Pi_{n-1}} \mu(\hat{0}, \sigma) \prod_{A \in \sigma} (t + n - 2 + \min(A)). \end{aligned}$$

The preceding discussion naturally raises the question of finding a similar interpretation for characteristic polynomials of extended Linial arrangements. As we shall show, Linial boards are crucial to this end. The route we follow is as follows: We express $r_{n-1}(L_{t,n})$ as the factorial polynomial of $L_{0,n}$ and observe that this polynomial has degree $n - 1$ in t . Using a weight-preserving bijection relating rook placements and labeled plane k -ary trees described in the next section, we compute $r_{n-1}(L_{t,n})$ for infinitely many integer specializations of t and then relate it to the characteristic polynomial of extended Linial arrangement.

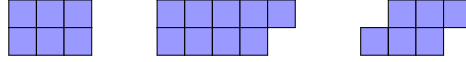


FIGURE 4. The t -Catalan, t -Shi and t -Linial boards for $n = 3$ and $t = 2$.

4. LABELED TREES AND ASCENT-DESCENT STATISTICS

A *rooted tree* is a finite acyclic graph with a distinguished vertex called the *root*. A *labeled rooted tree* on n vertices is a rooted tree whose vertices are labeled with distinct positive integers. A *rooted plane tree* is a rooted tree together with an ordering on the children of every vertex. Throughout, we consider the children to be ordered from left to right. A vertex in a rooted plane tree is a *leaf* if it has no children. Otherwise, it is called a *node*. A *rooted plane k -ary tree* is a rooted plane tree such that every node has k children. A *labeled rooted plane tree* on n nodes is a rooted plane tree whose nodes are labeled with distinct positive integers. In particular, note that leaves are not labeled. We refer to the label on a node v by v^ℓ . We denote the set of labeled rooted plane k -ary trees on n nodes with distinct labels drawn from $[n]$ by $\mathcal{PT}_{n,k}$. Given a node v of $T \in \mathcal{PT}_{n,k}$, we define the i -th *child* of v to be the i -th child from the left for $1 \leq i \leq k$. For all our trees, we implicitly assume that the edges are directed towards the root.

Given nodes p and q in $T \in \mathcal{PT}_{n,k}$ such that q is the i -th child of p , we call the edge from q to p an i -*descent* if $p^\ell > q^\ell$, and an i -*ascent* if $p^\ell < q^\ell$. We are interested in the distribution of ascent-descent statistics over $\mathcal{PT}_{n,k}$. To this end, with $T \in \mathcal{PT}_{n,k}$ we associate weak compositions

$$\begin{aligned} \text{des}(T) &= (\alpha_1, \dots, \alpha_k), \\ \text{asc}(T) &= (\beta_1, \dots, \beta_k), \end{aligned}$$

where α_i (respectively β_i) counts the number of i -descents (respectively i -ascents) in T . Note that $|\alpha| + |\beta| = n - 1$. Associate the weight $u^{\text{des}(T)} v^{\text{asc}(T)} := u_1^{\alpha_1} \cdots u_k^{\alpha_k} v_1^{\beta_1} \cdots v_k^{\beta_k}$ with T . For the labeled rooted plane

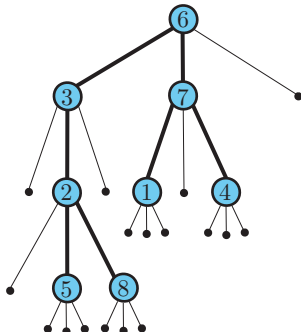


FIGURE 5. A labeled plane 3-ary tree with edges connecting nodes in bold.

ternary tree T in Figure 5, we have $\text{des}(T) = (2, 1, 1)$ and $\text{asc}(T) = (0, 2, 1)$. Therefore, its weight is $u_1^2 u_2^1 u_1^1 v_2^2 v_3^1$. Associate a polynomial $D_{n,k}(u, v)$ with $\mathcal{PT}_{n,k}$ as follows:

$$D_{n,k}(u, v) = \sum_{T \in \mathcal{PT}_{n,k}} u^{\text{des}(T)} v^{\text{asc}(T)}.$$

Note that $D_{n,1}$ tracks classical ascent-descent statistics over permutations of n letters. Drawing inspiration from a classic result of MacMahon [25] that excedance and descent statistics are equidistributed over permutations of n letters, we will establish that $E_{n,k} = D_{n,k}$ by constructing a weight-preserving bijection between $\tilde{\mathcal{F}}_{n,k}$ and $\mathcal{PT}_{n,k}$. Our algorithm is a mild generalization of the bijection in [11].

4.1. Bijection mapping (exc, sub) to (des, asc). We begin by providing a brief overview of our bijection $\Psi : \tilde{\mathcal{F}}_{n,k} \rightarrow \mathcal{PT}_{n,k}$. We take an element $f \in \tilde{\mathcal{F}}_{n,k}$ as input and construct its functional digraph $\vec{G}(f)$. After redirecting certain labeled edges in $\vec{G}(f)$, we obtain a labeled rooted (non-plane) tree \vec{T} whose edges carry labels drawn from $[k]$. Using the edge labels, we transform \vec{T} into a labeled rooted plane k -ary tree $\Psi(f)$. The formal description follows.

Let $f \in \tilde{\mathcal{F}}_{n,k}$. Construct the functional digraph $\vec{G}(f)$ corresponding to f as follows: The vertex set of $\vec{G}(f)$ consists of n vertices labeled from 1 through n . We identify vertices with their labels as this does not result in any confusion. For $1 \leq i \leq n - 1$, if $f(i) = (a, b)$, then we draw a directed edge labeled b from i to a . As f is an injective function from $[n - 1]$ into $[n] \times [k]$, it follows that there is no other edge labeled b directed towards a . Let $G(f)$ denote the undirected graph underlying $\vec{G}(f)$. For any connected component C of $G(f)$, we refer to the induced subgraph in $\vec{G}(f)$ whose vertex set is that of C as a *component* of $\vec{G}(f)$. For the f in Table 1, $\vec{G}(f)$ is shown in Figure 6. Note that there are 4 components in this instance.

Observe that, in general, $\vec{G}(f)$ has the following features.

- The component containing n is a labeled tree rooted at n .
- The other components (if any) contain a unique directed cycle (with length ≥ 1) such that deleting all edges participating in this cycle breaks the component into a set of labeled trees each of which is rooted at a vertex participating in the cycle. In Figure 6, the edges of these unique directed cycles in components of $\vec{G}(f)$ are highlighted.

Let C_1, \dots, C_r be the components of $\vec{G}(f)$ where we assume that n belongs to C_r . Let $\max(C_i)$ denote the maximum amongst all labels in the unique directed cycle in C_i for $1 \leq i \leq r - 1$. Also, let $\max(C_r) = n$. We further assume that $\max(C_i) < \max(C_{i+1})$ for $1 \leq i \leq r - 1$ as this can be ensured by the reordering C_1, \dots, C_{r-1} . In the functional digraph depicted in Figure 6, the four components C_1, C_2, C_3 and C_4 appear

from left to right and the $\max(C_i)$ for $i = 1, \dots, 4$ are 5, 7, 20 and 21 respectively. We turn $\vec{G}(f)$ into a labeled rooted tree as described next.

For $1 \leq i \leq r - 1$, suppose $f(\max(C_i)) = (s_i, t_i)$. For $1 \leq i \leq r - 1$, remove the directed edge labeled t_i connecting $\max(C_i)$ to s_i in $\vec{G}(f)$ and add a directed edge labeled t_i from $\max(C_{i+1})$ to s_i . The resulting digraph has a single component and the property that every vertex except $\max(C_1)$ has outdegree 1. Clearly, $\max(C_1)$ has outdegree 0. Thus, we may naturally identify this digraph with an edge-labeled tree \vec{T} rooted at $\max(C_1)$. For the $\vec{G}(f)$ in Figure 6, the corresponding \vec{T} (drawn rotated) is shown in Figure 7. The dashed edges are those that get redirected and the vertex labeled 5 is the root.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$f(i)$	3_1	5_2	4_3	5_3	3_3	21_2	7_1	12_2	1_3	4_2	4_1	20_2	19_3	19_1	6_2	1_2	16_1	6_3	7_2	12_3

TABLE 1. An element of $\vec{\mathcal{F}}_{21,3}$. For the sake of convenience, by $f(i) = a_b$ we mean $f(i) = (a, b)$.

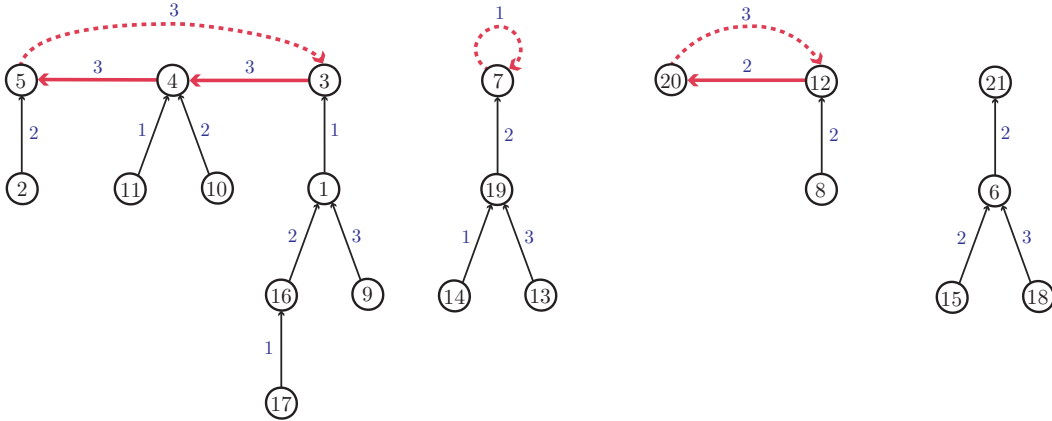


FIGURE 6. Functional digraph corresponding to f from Table 1.

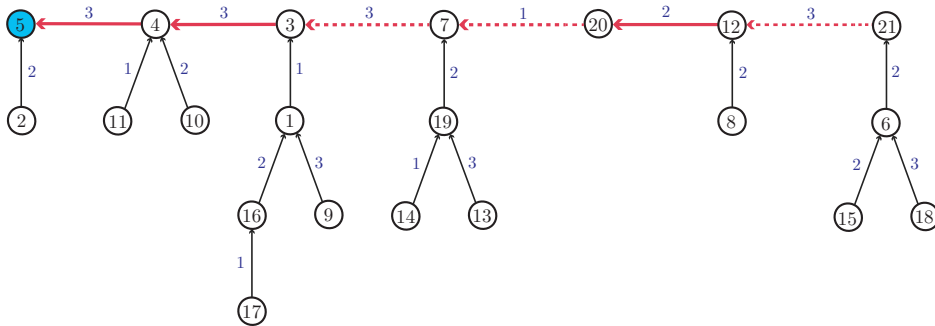


FIGURE 7. The resulting labeled rooted tree after redirecting dashed edges in $\vec{G}(f)$.

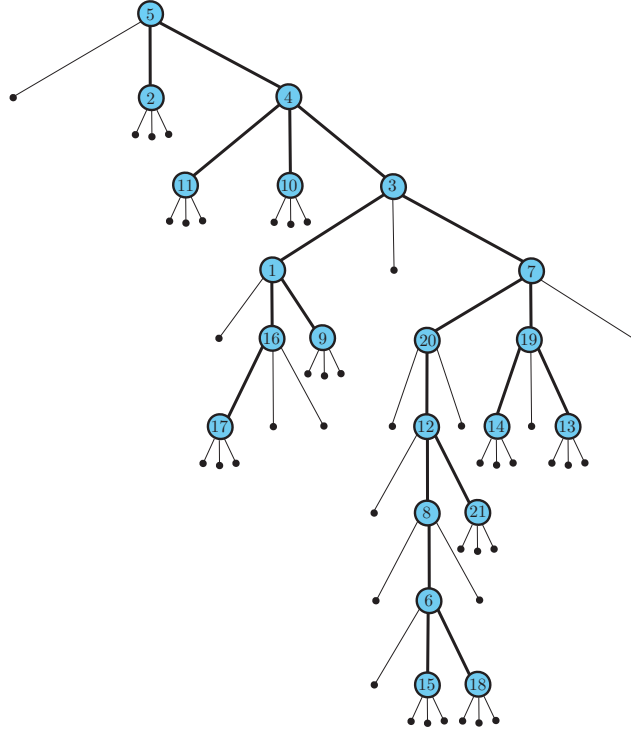


FIGURE 8. The labeled rooted plane ternary tree $\Psi(f)$ corresponding to f in Table 1.

Note that the procedure just outlined is reversible. We may uniquely recover $\vec{G}(f)$ from \vec{T} as follows: Consider the path from root to the vertex labeled n in \vec{T} . Let the sequence of labels encountered on this path be $L := l_1 \dots l_m$. We call $1 \leq i \leq m$ a *record* of L if $l_i > l_j$ for all $1 \leq j < i$. In particular, 1 and m are both records of L . Suppose p and q are distinct records of L such that no r satisfying $p < r < q$ is a record. That is, p and q are *successive* records. Let the label of the edge in \vec{T} from the vertex labeled l_q to the vertex labeled l_{q-1} be c . Remove this edge and, instead, add a directed edge labeled c from the vertex labeled l_p to the vertex labeled l_{q-1} . Repeating this procedure for every pair of successive records recovers $\vec{G}(f)$ from \vec{T} . For \vec{T} in Figure 7, we have $L = 5, 4, 3, 7, 20, 12, 21$, and the records of L are 1, 4, 5 and 7. So, for instance, the labels on the vertices corresponding to the successive records 4 and 5 are 7 and 20 respectively and the edge joining them has label 1. To recover $\vec{G}(f)$, we remove this edge and instead insert an edge labeled 1 from 7 to itself.

Remark 4.1. *Observe that our records are a straightforward generalization of records (or left-to-right maxima) of a permutation.*

At this point one may easily transform \vec{T} to a labeled rooted *plane* k -ary tree as follows: If the edge directed from vertex p to vertex q has label i , then set p to be the i -th child of q . Complete the tree by adding additional leaves if necessary. We denote the element of $\mathcal{PT}_{n,k}$ thus obtained by $\Psi(f)$. Figure 8 shows the labeled rooted plane ternary tree corresponding to \vec{T} in Figure 7. The edges connecting nodes are in bold.

From the preceding discussion, it follows that Ψ is a bijection between $\tilde{\mathcal{F}}_{n,k}$ and $\mathcal{PT}_{n,k}$ and, therefore, induces a bijection between $\mathcal{F}_{n,k}$ and $\mathcal{PT}_{n,k}$. Our next theorem relates the ascent-descent statistics on $\mathcal{PT}_{n,k}$ to excedance-subcedance statistics on $\tilde{\mathcal{F}}_{n,k}$.

Theorem 4.2. $\Psi : \widetilde{\mathcal{F}}_{n,k} \rightarrow \mathcal{PT}_{n,k}$ is a bijection such that $\text{exc}(f) = \text{des}(\Psi(f))$ and $\text{sub}(f) = \text{asc}(\Psi(f))$.

Proof. Throughout this proof, we adhere to notation established in the preceding description of Ψ . We have already shown that Ψ is a bijection and therefore focus on its weight-preserving aspect. Let $f \in \widetilde{\mathcal{F}}_{n,k}$ and $T := \Psi(f)$. Direct the edges of T towards its root. Let $\text{sub}(f) = (\beta_1, \dots, \beta_k)$ and $\text{asc}(T) = (\gamma_1, \dots, \gamma_k)$. We show that $\text{asc}(T) = \text{sub}(f)$ and omit the argument for $\text{des}(T) = \text{exc}(f)$ as it is similar.

Fix $1 \leq i \leq k$. Note that γ_i equals the number of edges connecting nodes b to a in T such that $b^\ell > a^\ell$ and b is the i -th child of a . Similarly, β_i equals the number of edges labeled i directed from vertices b to a in $\vec{G}(f)$ such that the label of b is strictly greater than the label of a . Consider the sequence of labels $L := l_1 \dots l_m$ in T along the unique path from root to the node labeled n . Suppose that the corresponding nodes in T are v_1, \dots, v_m . Let $1 = s_1 < s_2 < \dots < s_p = m$ be the records of L . We consider the procedure of recovering $\vec{G}(f)$ from T . Let $S = \{j \mid v_{s_j} \text{ is the } i\text{-th child of } v_{s_{j-1}} \text{ in } T\}$. Note that $1 \notin S$. Assume S is nonempty and consider $q \in S$. Since s_q is a record, we have $l_{s_q} > l_{s_{q-1}}$ and the edge connecting v_{s_q} to $v_{s_{q-1}}$ is an i -ascent, and therefore contributes to γ_i . To obtain $\vec{G}(f)$, we redirect this edge to point from $v_{s_{q-1}}$ to v_{s_q} and endow it the label i . As s_q and s_{q-1} are successive records, we have $l_{s_{q-1}} \geq l_{s_q}$. As the label of this edge is i , it contributes to β_i . Since edges in T that do not lie on the path from root to node labeled n are unaltered in obtaining $\vec{G}(f)$ from T , we infer that $\gamma_i = \beta_i$ and therefore $\text{asc}(T) = \text{sub}(f)$. \square

As an immediate corollary of Theorem 4.2, we obtain our first main result, which may be considered a k -ary generalization of the classical result of MacMahon on the equidistribution of descent and excedance statistics over the set of permutations on n letters.

Corollary 4.3. Given positive integers n and k , we have $E_{n,k} = D_{n,k}$.

In view of Corollary 4.3, we define $G_{n,k} := E_{n,k} = D_{n,k}$ to be the *Gessel polynomial* indexed by positive integers n and k . The polynomials $G_{n,2}$ were introduced by Gessel [13] and studied further by Drake [10] and Kalikow [21]. More precisely, they considered the generating function

$$B(u_1, u_2, v_1, v_2; x) = \sum_{n \geq 1} G_{n,2}(u_1, u_2, v_1, v_2) \frac{x^n}{n!}$$

and established that $B := B(u_1, u_2, v_1, v_2; x)$ satisfies the functional equation:

$$\frac{(1 + v_1 B)(1 + u_2 B)}{(1 + v_2 B)(1 + u_1 B)} = e^{[(v_1 u_2 - v_2 u_1)B + v_1 - v_2 - u_1 + u_2]x}.$$

Note further that, from classical results on rooted labeled binary trees, we know that $G_{n,2}(1, 1, 1, 1) = n!C_n$ where $C_n = \frac{1}{n+1} \binom{2n}{n}$ denotes the n -th Catalan number. Additionally, $G_{n,2}(1, 1, 1, 0) = (n+1)^{n-1}$, $G_{n,2}(1, 0, 1, 0) = n!$ and $G_{n,2}(u_1, u_2, 0, 0) = \sum_{\sigma \in \mathfrak{S}_n} u_1^{\text{des}(\sigma)} u_2^{\text{asc}(\sigma)}$ (the n -th homogenized Eulerian polynomial). The careful reader might recall that the number of regions in the Catalan arrangement is $n!C_n$, the number of regions in the Shi arrangement is $(n+1)^{n-1}$, and the number of regions in the braid arrangement is $n!$. The specialization $G_{n,2}(1, 0, 0, 1)$ is equally interesting and enumerates regions of the Linial arrangement and it is a generalization of this evaluation that we study in some depth.

5. REGIONS IN EXTENDED LINIAL ARRANGEMENTS VIA GESSEL POLYNOMIALS

From this point on, given a node v in $T \in \mathcal{PT}_{n,k}$, we refer to the first child of v as its *left child* and the k -th child of v as its *right child*. As we want to distinguish between left and right children, we assume further that $k \geq 2$. Consider the following subsets of $\mathcal{PT}_{n,k}$.

$$\mathcal{L}_{n,k} = \{T \in \mathcal{PT}_{n,k} \mid T \text{ has no 1-ascents or } k\text{-descents}\},$$

$$\mathcal{L}_{n,k}^b = \{T \in \mathcal{L}_{n,k} \mid \text{the node labeled 1 (respectively } n) \text{ has a leaf as its right child (respectively left child)}\}.$$

Figure 9 depicts all trees in $\mathcal{L}_{4,2}^b$. In this section we relate cardinalities of the aforementioned sets to regions

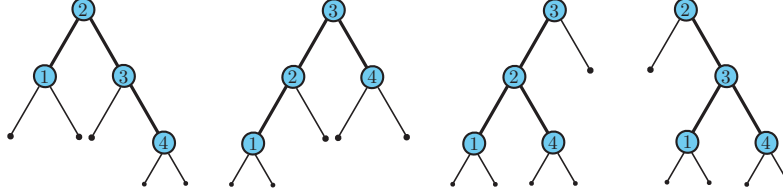


FIGURE 9. The four labeled rooted plane binary trees in $\mathcal{L}_{4,2}^b$.

in Linial arrangements using a beautiful result of Drake [10] that allows us to compute the exponential generating function for ascent-descent statistics on $\mathcal{PT}_{n,k}$. Let

$$P_k(\mathbf{u}, \mathbf{v}; x) := \sum_{n \geq 1} \sum_{T \in \mathcal{PT}_{n,k}} \mathbf{u}^{\text{des}(T)} \mathbf{v}^{\text{asc}(T)} \frac{x^n}{n!} = \sum_{n \geq 1} G_{n,k} \frac{x^n}{n!}.$$

The following result of Drake [10, Theorem 1.8.4] expresses $P_k := P_k(\mathbf{u}, \mathbf{v}; x)$ as the compositional inverse of another function.

Theorem 5.1 ([10]). *Let $Z(a, b, c, d) = (ad - bc)x + (a - c) - (b - d)$. Then*

$$P_k = \left(\sum_{i=1}^k (v_i - u_i)^{k-2} \log \left(\frac{1 + v_i x}{1 + u_i x} \right) \prod_{j \in [k], i \neq j} \frac{1}{Z(v_i, u_i, v_j, u_j)} \right)^{\langle -1 \rangle}.$$

We use Theorem 5.1 to compute $|\mathcal{L}_{n,k}|$, denoted by $l_{n,k}$, when $k \geq 2$. As the $l_{n,k}$ enumerate trees in $\mathcal{PT}_{n,k}$ that do not possess 1-ascents or k -descents, we obtain the exponential generating function for these numbers by evaluating P_k at $u_k = v_1 = 0$ and all the other variables set to 1. Consider

$$\tilde{P}_k = \sum_{n \geq 1} l_{n,k} \frac{x^n}{n!}.$$

Theorem 5.1 implies the following functional equation for \tilde{P}_k . The details are similar to the numerous examples in [10, Chapter 1.9] and are left to the reader.

$$\tilde{P}_k = \left(\frac{2 \log(1 + x)}{(1 + x)^{k-2} (2 + x)} \right)^{\langle -1 \rangle}.$$

Substituting $f = 1 + \tilde{P}_k$, we get that f satisfies the relation

$$(5.1) \quad f^2 = e^{x(f^{k-2} + f^{k-1})}.$$

The functional equation in Equation 5.1 is precisely the one in the statement of [28, Theorem 9.3] with $b = k$ and $a = k - 2$. It follows that $l_{n,k}$, which is the coefficient of $\frac{x^n}{n!}$ in f for $n \geq 1$, is given by

$$(5.2) \quad l_{n,k} = \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (1 + (k - 2)n + j)^{n-1}.$$

Remark 5.2. *In the case $k = 2$, the $l_{n,k}$ enumerate local binary search trees on n nodes, and the formula obtained above coincides with that obtained by Postnikov [26] and Athanasiadis [2, Theorem 4.2]. The trees in $\mathcal{L}_{n,k}$ are clearly a natural generalization of local binary search trees.*

We turn our attention towards computing $|\mathcal{L}_{n,k}^b|$ and then relating it to the number of bounded regions in extended Linial arrangements. Our bijection Ψ is crucial for this purpose.

5.1. Trees in $\mathcal{L}_{n,k}$ and rook placements. Consider the Linial board $L_{t,n}$ for $t = (a-1)n + 2$ where a is a positive integer. Maximal nonattacking rook placements on $L_{(a-1)n+2,n}$ are in bijection with the subset of $\mathcal{F}_{n,a+1}$ that consists of injective functions $f : [n-1] \rightarrow [(a+1)n]$ satisfying $i+1 \leq f(i) \leq an+i$ for $1 \leq i \leq n-1$. Our bijection Ψ from Section 4.1 implies that the number of such rook placements is the number of trees in $T \in \mathcal{PT}_{n,a+1}$ that have neither 1-ascents and nor $(a+1)$ -descents. Indeed, the condition $i+1 \leq f(i)$ implies that T has no 1-ascents whereas the condition $f(i) \leq an+i$ implies that T has no $(a+1)$ -descents. This motivates our next theorem that gives a closed form expression for the factorial polynomial of the Linial board $L_{0,n}$.

Proposition 5.3. *We have that*

$$R(t, L_{0,n}) = \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (t-1+j)^{n-1}.$$

Proof. Since $L_{0,n}$ has $n-1$ rows, its factorial polynomial $R(t, L_{0,n})$ has degree $n-1$. The definition of factorial polynomial further implies that the evaluations of $R(t, L_{0,n})$ at $t = m$ for m a nonnegative integer agree with $r_{n-1}(L_{m,n})$. By comparison with Equation (5.2), we know that

$$r_{n-1}(L_{m,n}) = \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (1+(m-2)+j)^{n-1}$$

when m is of the form $(a-1)n + 2$ for a positive integer $a \geq 1$. Thus, the polynomial of degree $n-1$:

$$\frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} (1+(t-2)+j)^{n-1},$$

agrees with $R(t, L_{0,n})$ for infinitely many integer values t . Therefore the two must be equal. \square

Given the similarity between the expression in Proposition 5.3 and in [28, Equation 9.11][2, Theorem 4.2], the next result is natural.

Theorem 5.4. *Given a positive integer a , the following hold:*

$$\begin{aligned} (-1)^{n-1} \chi_{n-1}^{a-1, a+1}(q) &= R(1+(a-1)n-q, L_{0,n}), \\ (-1)^{n-1} \chi_{n-1}^{a-1, a+1}(q) &= \sum_{\sigma \in \Pi_{n-1}} \mu(\hat{0}, \sigma) \prod_{A \in \sigma} (an-q-1+\min(A)-\max(A)). \end{aligned}$$

Proof. The first equality follows by comparing the expression for $\chi_{n-1}^{a-1, a+1}$ in [28, Equation 9.11] to the expression for $R(t, L_{0,n})$ in Proposition 5.3. For the second equality, we apply Theorem 3.1 to compute $R(t, L_{0,n})$ and use the fact that $v_{L_{0,n}}(A)$ is equal to $n-2-(\max(A)-\min(A))$ for $A \subseteq [n-1]$. \square

We are ready to prove our main enumerative result, a special case of which was presented as Theorem 1.1.

Corollary 5.5. *For a fixed positive integer a , let $\mathcal{A} := \text{Lin}_{n-1, a} = \mathcal{A}_{n-1}^{a-1, a+1}$. Then the following hold.*

- (1) $r(\mathcal{A}) = R((a-1)n+2, L_{0,n})$ and $b(\mathcal{A}) = R((a-1)n, L_{0,n})$.
- (2) $r(\mathcal{A}) = r_{n-1}(L_{(a-1)n+2,n})$ and $b(\mathcal{A}) = r_{n-1}(L_{(a-1)n,n})$.
- (3) $r(\mathcal{A}) = |\mathcal{L}_{n,a+1}|$ and $b(\mathcal{A}) = |\mathcal{L}_{n,a+1}^b|$.
- (4) $r(\mathcal{A})$ equals the number of sequences (x_1, \dots, x_{n-1}) such that $1 \leq x_i \leq an$ for $1 \leq i \leq n-1$ and $x_i + i \neq x_j + j$ for $1 \leq i \neq j \leq n-1$; and $b(\mathcal{A})$ equals the number of sequences (x_1, \dots, x_{n-1}) such that $1 \leq x_i \leq an-2$ for $1 \leq i \leq n-1$ and $x_i + i \neq x_j + j$ for $1 \leq i \neq j \leq n-1$.

Proof. The first claim follows from plugging $q = \pm 1$ in Theorem 5.4 and recalling Zaslavsky’s result, that is, Theorem 2.1. The second claim follows from the first by using the relation between evaluations of factorial polynomial and maximal nonattacking rook placements.

The discussion preceding Theorem 5.3 establishes that $r_{n-1}(L_{(a-1)n+2,n}) = |\mathcal{L}_{n,a+1}|$ implying the first part of the third claim. As for the second part, note that maximal nonattacking rook placements on $L_{(a-1)n,n}$ can be thought of maximal nonattacking rook placements on $L_{(a-1)n+2,n}$ where we forbid the rooks to belong to columns n and $an+1$.[†] In light of our bijection, such rook placements are in bijection with trees in $\mathcal{L}_{n,a+1}$ which satisfy the criterion that the node labeled 1 (respectively n) has a leaf as its right child (respectively left child). These are precisely trees that belong to $\mathcal{L}_{n,a+1}^b$. This establishes the third claim.

The final claim follows easily by left-justifying our Linial boards. The columns occupied by the rooks in the left-justified board give us our sequence (x_1, \dots, x_{n-1}) (reading the rows from bottom to top) and the condition $x_i + i \neq x_j + j$ is tantamount to the nonattacking condition. \square

5.2. Chromatic polynomials and graphs from Linial boards. Recall that characteristic polynomials of graphic arrangements are chromatic polynomials of their associated graphs. In particular, the characteristic polynomial of the braid arrangement is the chromatic polynomial of the complete graph. We briefly consider the question of finding an analogue of the preceding result in the context of extended Linial arrangements.

We begin by associating a bipartite graph G_n with the Linial board $L_{0,n}$ for $n \geq 2$. Consider a set of vertices $V = \{v_1, \dots, v_{2n-4}\}$ where v_i represents the i th column from the left in $L_{0,n}$ for $1 \leq i \leq 2n-4$. Similar, consider another set of vertices $W = \{w_1, \dots, w_{n-1}\}$ where w_i represents the i th row from the bottom in $L_{0,n}$ for $1 \leq i \leq n-1$. This given, G_n is the bipartite graph with vertex set $V \cup W$ with an edge between w_i and v_j for every (i, j) in $L_{0,n}$. We denote the complement of G_n by $\overline{G_n}$. Figure 10 depicts the board $L_{0,3}$ and the graph G_3 with bold edges. The dashed edges induce the complement $\overline{G_3}$.

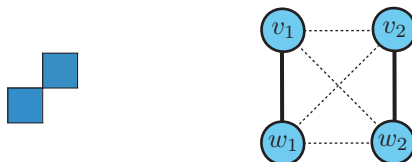


FIGURE 10. The board $L_{0,3}$ on the left and the graph G_3 (with bold edges) on the right.

It is clear that nonattacking rook placements on $L_{0,n}$ can be interpreted as matchings on G_n . More interestingly, we claim that the chromatic polynomial of $\overline{G_n}$ is the $(3n-5)$ -factorial polynomial of $L_{0,n}$. We explain this briefly next. Note that $3n-5$ is the number of vertices in $\overline{G_n}$. Furthermore, up to an interchange of rows and columns, G_n is obtained from a *proper* $(3n-5)$ -board. The claim follows from invoking [15, Theorem 2]. We refer the reader to [15] for the definition of proper boards and more details on the relation between factorial polynomials of proper boards and chromatic polynomials of associated graphs.

Theorem 5.6. *Denote the chromatic polynomial of $\overline{G_n}$ by $c(x, \overline{G_n})$. Let a be a fixed positive integer. Then*

$$\frac{1}{(x)_{2n-4}} c(x, \overline{G_n}) = (-1)^{n-1} \chi_{n-1}^{a-1, a+1} ((a+1)n - 3 - x).$$

Proof. By [15, Theorem 2] we have $c(x, \overline{G_n}) = R_{3n-5}(x, L_{0,n})$. This is equivalent to $c(x, \overline{G_n}) = (x)_{2n-4} R(x - 2n + 4, L_{0,n})$. From Theorem 5.4, we have $R(x - 2n + 4, L_{0,n}) = (-1)^{n-1} \chi_{n-1}^{a-1, a+1} ((a+1)n - 3 - x)$, thereby establishing the claim. \square

[†]One may forbid any two columns of length $n-1$ in fact. Our choice leads to a nicer combinatorial interpretation for the number of bounded regions in extended Linial arrangements.

Thus, up to a certain normalizing factor, the chromatic polynomials of the graphs $\overline{G_n}$ equal the characteristic polynomials of extended Linial arrangements. For the graph G_3 in Figure 10, we have $c(x, \overline{G_3}) = x(x-1)(x^2-3x+3)$ and $\chi_2^{0,2}(x) = x^2-3x+3 = \chi_2^{0,2}(3-x)$. Clearly, $\frac{1}{(x)_2}c(x, \overline{G_3}) = \chi_2^{0,2}(3-x)$.

6. FINAL REMARKS

We close with some remarks about future avenues and related work.

- (1) It is natural to ask if one can extend our results to the entire class of truncated affine arrangements; or even to subarrangements thereof. The Shi(G) arrangements [2, 6] and the G -Shi arrangements [19] have garnered plenty of attention recently and remain an active area of research. A rook-theoretic perspective on them might be of interest and lend new insight. Furthermore, it would be interesting to obtain q -analogs of our formulae and extend the results of Haglund-Remmel [17] to our skew boards. Finally, it might be worthwhile to explore the relation to work of Ardila on Tutte polynomials associated with hyperplane arrangements [1].
- (2) In this article, we expressed characteristic polynomials of the extended Catalan, extended Shi and extended Linial arrangements as certain sums over the lattice of set partitions. It is interesting to note that similar results were achieved in [8] from a different perspective, and the equivalence with our results is not obvious.
- (3) We will focus on the case $k = 2$ of our bijection. We know that the number of maximal nonattacking rook placements on the skew Ferrers board $(2n-1, \dots, n+1)/(n-1, \dots, 1)$ equals the number of local binary search trees on $[n]$. Clearly, the set of such rook placements is equinumerous with the set of bijections $\{f : \mathbb{Z} \mapsto \mathbb{Z}\}$ satisfying the following conditions:
 - (a) $f(i+n-1) = f(i) + (n-1)$,
 - (b) $i \leq f(i) \leq i+n-1$.
 It is worth remarking that this is a subset of conditions used to define *bounded affine permutations* [22] and we wonder if there is a stronger connection. Also, nonattacking rook placements on skew Ferrers boards have also been studied in [27, 34] and it might be interesting to put our results in the context of the aforementioned articles.
- (4) If we consider the rook placements corresponding to increasing labeled rooted k -ary trees on n nodes, we are essentially counting maximal nonattacking rook placements on the Ferrers board of shape $(k(n-1), k(n-2), \dots, k)$. Josuat-Vergès [20] studied the case $k = 2$ using stammering tableaux and growth diagrams. It would be interesting to explore the case $k > 2$.

Finally, we would be remiss to not mention other work inspired by Gessel's original question, namely [9, 12], and more recently [7] which answers Gessel's questions from a different perspective than ours. Connections with the aforementioned work, especially [7], and ours remain to be explored.

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