A MURNAGHAN-NAKAYAMA RULE FOR NONCOMMUTATIVE SCHUR FUNCTIONS

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Abstract. We prove a Murnaghan-Nakayama rule for noncommutative Schur functions introduced by Bessenrodt, Luoto and van Willigenburg. In other words, we give an explicit combinatorial formula for expanding the product of a noncommutative power sum symmetric function and a noncommutative Schur function in terms of noncommutative Schur functions. In direct analogy to the classical Murnaghan-Nakayama rule, the summands are computed using a noncommutative analogue of border strips, and have coefficients ± 1 determined by the height of these border strips. The rule is proved by interpreting the noncommutative Pieri rules for noncommutative Schur functions in terms of box-adding operators on compositions.

Contents

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1. Introduction

The Murnaghan-Nakayama rule [11, 12] is a combinatorial procedure to compute the character table of the symmetric group. The combinatorial objects involved, that aid the said computation, are called border strips. An alternate description of the Murnaghan-Nakayama rule is that it gives an explicit combinatorial description of the expansion of the product of a power sum symmetric function p_k, where k is a positive integer, and a Schur function s_μ as a sum of Schur functions in the algebra of symmetric functions

$$p_k \cdot s_\mu = \sum_{\lambda} (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda, \tag{1}$$

where the sum is over all partitions λ such that the skew shape λ/μ is a border strip of size k and $\mathrm{ht}(\lambda/\mu)$ is a certain statistic associated with the border strip λ/μ. We can iterate the above rule to expand a power sum symmetric function indexed by a partition in terms of Schur functions, and the structure coefficients thus obtained are character values of the symmetric group.

In this article, we consider an analogue in the algebra of noncommutative symmetric functions. The role played by Schur functions in the classical setting is taken by noncommutative Schur functions introduced in [2], that are dual to quasisymmetric Schur functions [6] arising from the combinatorics of Macdonald polynomials [5]. To be more precise, we expand the product $\Psi_r \cdot \mathbf{s}_\alpha$ in the basis of noncommutative Schur functions. Here Ψ_r denotes the noncommutative power sum symmetric function of the first kind, indexed by a positive integer r, while $\mathbf{s}_\alpha$ denotes the noncommutative Schur function indexed by a composition α. The main theorem we prove is that, much like (1), we have an expansion

$$\Psi_r \cdot \mathbf{s}_\alpha = \sum_{\beta} (-1)^{\mathrm{ht}(\beta /\!\!/ \alpha)} \mathbf{s}_\beta, \tag{2}$$

where the sum is over certain compositions β such that the skew reverse composition shape $\beta /\!\!/ \alpha$ gives rise to a noncommutative analogue of a border strip of size r and $\mathrm{ht}(\beta /\!\!/ \alpha)$ is a certain statistic similar to $\mathrm{ht}(\mu/\lambda)$ in (1).

The organization of this article is as follows. In Section 2, we introduce all the notation and definitions necessary for stating the classical Murnaghan-Nakayama rule. Section 3 introduces the algebra of noncommutative symmetric functions and the distinguished basis of noncommutative Schur functions amongst other bases. Furthermore, we introduce box-adding operators reminiscent of those in [1, 3] that will be fundamental to our proofs. The goal of Section 4 is to give a rudimentary version of the noncommutative Murnaghan-Nakayama rule in terms of box-adding operators in Equation (10). In Section 5, we give the noncommutative Murnaghan-Nakayama rule that mirrors the classical version in Theorem 5.16. Finally, in Section 6 we give a reformulation of our rule in Theorem 6.14. The approach in this section is algorithmic, with an eye towards computing the product efficiently. We also discuss how the classical Murnaghan-Nakayama rule follows from ours.

2. Background on symmetric functions

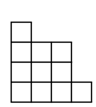
2.1. **Partitions.** We will start by defining some of the combinatorial structures that we will be encountering. All the notions introduced in this section are covered in more detail in [10, 14, 13].

Definition 2.1. *A* partition λ *is a finite list of positive integers* $(\lambda_1, \dots, \lambda_k)$ *satisfying* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. *The integers appearing in the list are called the* parts *of the partition.*

Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, the *size* $|\lambda|$ is defined to be $\sum_{i=1}^{k} \lambda_i$. The number of parts of λ is called the *length*, and is denoted by $l(\lambda)$. If λ is a partition satisfying $|\lambda| = n$, then we write it as $\lambda \vdash n$. By convention, there is a unique partition of size and length 0, and we denote it by $\varnothing$.

We will be depicting a partition using its *Young diagram*. For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of size n, the Young diagram of λ, also denoted by λ, is the left-justified array of n boxes, with λ_i boxes in the i-th row. We will be using the French convention, that is, the rows are numbered from bottom to top and the columns from left to right. We refer to the box in the i-th row and j-th column by the ordered pair (i, j). If λ and μ are partitions such that $\mu \subseteq \lambda$, that is, $l(\mu) \leq l(\lambda)$ and $\mu_i \leq \lambda_i$ for all $i = 1, 2, \dots, l(\mu)$, then the *skew shape* λ/μ is obtained by removing the first μ_i boxes from the i-th row of the Young diagram of λ for $1 \leq i \leq l(\mu)$. The *size* of the skew shape λ/μ is equal to the number of boxes in the skew shape, that is, $|\lambda| - |\mu|$.

Example 2.2. The Young diagram for the partition $\lambda = (4, 3, 3, 1) \vdash 11$ is shown below.



2.2. **Semistandard reverse tableaux.** In this subsection, we will introduce some classical objects that play a central role in the theory of symmetric functions.

Definition 2.3. *Given a partition λ, a* semistandard reverse tableau (SSRT) T *of* shape λ *is a filling of the boxes of λ with positive integers, satisfying the condition that the entries in T are weakly decreasing along each row read from left to right and strictly decreasing along each column read from bottom to top.*

A *standard reverse tableau (SRT)* T of shape $\lambda \vdash n$ is an SSRT that contains every positive integer in $[n] = \{1, 2, \ldots, n\}$ exactly once. We will denote the set of all SSRTs of shape λ by $SSRT(\lambda)$. Given an SSRT T, the entry in box (i, j) is denoted by $T_{(i,j)}$.

Let $\{x_1, x_2, \ldots\}$ be an alphabet comprising of countably many commuting indeterminates $x_1, x_2, \ldots$. Now, given any SSRT T of shape $\lambda \vdash n$, we can associate a monomial x^T with it as follows.

$$x^T = \prod_{(i,j)\in\lambda} x_{T_{(i,j)}}$$

2.3. **Symmetric functions.** The algebra of symmetric functions, denoted by Λ, is the algebra freely generated over $\mathbb{Q}$ by countably many commuting variables $\{h_1, h_2, \ldots\}$. Assigning degree i to h_i (and then extending this multiplicatively) allows us to endow Λ with a structure of a graded algebra. A basis for the degree n component of Λ, denoted by Λ^n, is given by the *complete homogeneous symmetric functions* of degree n,

$$\{h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k} \mid \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n\}.$$

A concrete realization of Λ is obtained by embedding $\Lambda = \mathbb{Q}[h_1, h_2, \ldots]$ in $\mathbb{Q}[[x_1, x_2, \ldots]]$ under the identification (extended multiplicatively)

$$h_i \longmapsto \text{ sum of all distinct monomials in } x_1, x_2, \ldots \text{ of degree } i.$$

This viewpoint allows us to think of symmetric functions as being formal power series f in the x variables with the property that $f(x_{\pi(1)}, x_{\pi(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation π of the positive integers $\mathbb{N}$.

The *Schur function* indexed by the partition λ, s_λ, is defined combinatorially as

$$s_\lambda = \sum_{T\in SSRT(\lambda)} x^T.$$

Though not evident from the definition above, s_λ is a symmetric function. Furthermore, the elements of the set $\{s_\lambda \mid \lambda \vdash n\}$ form a basis of Λ^n for any positive integer n.

Another important class of symmetric functions is given by *power sum symmetric functions*. The power sum symmetric function p_k for $k \geq 1$ is defined as

$$p_k = \sum_{i\geq 1} x_i^k.$$

The classical Murnaghan-Nakayama rule gives an algorithm to compute the product $p_k \cdot s_\mu$ in terms of Schur functions using the notion of a border strip. A skew shape λ/μ is called a *border strip* if it is connected and contains no 2×2 array of boxes. The *height* of a border strip λ/μ, denoted by $\mathrm{ht}(\lambda/\mu)$, is defined to be one less than the number of rows occupied by the border strip.

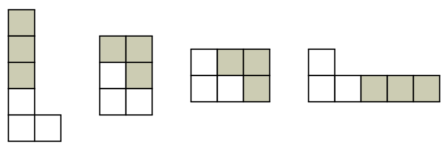
Theorem 2.4 (Murnaghan-Nakayama rule)**.** *Given a positive integer k and $\mu \vdash n$, we have*

$$p_k \cdot s_\mu \;=\; \sum_{\lambda \vdash |\mu|+k} (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda,$$

where the sum is over all partitions λ such that λ/μ is a border strip of size k.

As an aid to understanding the theorem above, we will do an example.

Example 2.5. Consider the computation of $p_3 \cdot s_{(2,1)}$. All partitions λ such that $\lambda/(2,1)$ is a border strip of size 3 are listed below.



The statistic $\mathrm{ht}(\lambda/(2,1))$ for the partitions above from left to right is $2, 1, 1, 0$ respectively. Hence, the Murnaghan-Nakayama rule implies that

$$p_3 \cdot s_{(2,1)} = s_{(2,1,1,1,1)} - s_{(2,2,2)} - s_{(3,3)} + s_{(5,1)}.$$

3. Background on noncommutative symmetric functions

3.1. **Compositions.**

Definition 3.1. *A* composition α *is a finite ordered list of positive integers. The integers appearing in the list are called the* parts *of the composition.*

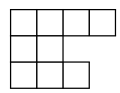
Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, the *size* $|\alpha|$ is defined to be $\sum_{i=1}^{k} \alpha_i$. The number of parts of α is called the *length*, and is denoted by $l(\alpha)$. If α is a composition satisfying $|\alpha| = n$, then we write it as $\alpha \vDash n$. By convention, there is a unique composition of size and length 0, and we denote it by $\varnothing$.

We will associate a *reverse composition diagram* to a composition as follows. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$, the *reverse composition diagram* of α, also denoted by α, is the left-justified array of n boxes with α_i boxes in the i-th row. Here we follow the English convention, that is, the rows are numbered from top to bottom, and the columns from left to right. Again, we refer to the box in the i-th row and j-th column by the ordered pair (i, j).

Recall now the bijection between compositions of n and subsets of $[n-1]$. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$, we can associate a subset of $[n-1]$, called $\mathrm{set}(\alpha)$, by defining it to be $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$. In the opposite direction, given a set $S = \{i_1 < \cdots < i_j\} \subseteq [n-1]$, we can associate a composition of n, called $\mathrm{comp}(S)$, by defining it to be $(i_1, i_2 - i_1, \ldots, i_j - i_{j-1}, n - i_j)$.

Finally, we define the refinement order on compositions. Given compositions α and β, we say that $\alpha \succcurlyeq \beta$ if one obtains parts of α in order by adding together adjacent parts of β in order. The composition β is said to be a *refinement* of α. For instance, $(3, 1, 2, 1, 2)$ is a refinement of $(4, 2, 3)$ and we denote this by $(4, 2, 3) \succcurlyeq (3, 1, 2, 1, 2)$.

Example 3.2. Let $\alpha = (4, 2, 3) \vDash 9$. Then $\mathrm{set}(\alpha) = \{4, 6\} \subseteq [8]$. Shown below is the reverse composition diagram of α.



3.2. **Semistandard reverse composition tableaux.** Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ and β be compositions. Define a cover relation, $\lessdot_c$, on compositions as follows.

$$\alpha \lessdot_c \beta \text{ iff } \begin{cases} \beta = (1, \alpha_1, \ldots, \alpha_l) & \text{or} \\ \beta = (\alpha_1, \ldots, \alpha_k + 1, \ldots, \alpha_l) & \text{and } \alpha_i \neq \alpha_k \text{ for all } i < k. \end{cases}$$

The *reverse composition poset* $\mathcal{L}_c$ is the poset on compositions where the partial order $<_c$ is obtained by taking the transitive closure of the cover relations above. If $\alpha <_c \beta$, the *skew reverse composition shape* $\beta /\!\!/ \alpha$ is defined to be the array of boxes

$$\beta /\!\!/ \alpha = \{(i, j) \mid (i, j) \in \beta, (i, j) \notin \alpha\}$$

where α is drawn in the bottom left corner of β. We refer to β as the *outer shape* and to α as the *inner shape*. If the inner shape is $\varnothing$, instead of writing $\beta /\!\!/ \varnothing$, we just write β and refer to β as a *straight shape*. The *size* of the skew reverse composition shape $\beta /\!\!/ \alpha$, denoted by $|\beta /\!\!/ \alpha|$, is the number of boxes in the skew reverse composition shape, that is, $|\beta| - |\alpha|$.

If $\beta /\!\!/ \alpha$ does not have two boxes belonging to the same column, then it is called a *horizontal strip*, while if it does not have two boxes lying in the same row it is called a *vertical strip*.

Definition 3.3. *A* semistandard reverse composition tableau *(SSRCT) τ of shape $\beta /\!\!/ \alpha$ is a filling*

$$\tau : \beta /\!\!/ \alpha \longrightarrow \mathbb{Z}^+$$

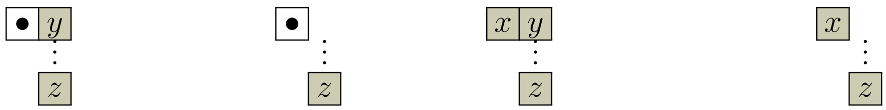
that satisfies the following conditions.

1. *The rows are weakly decreasing from left to right.*
2. *The entries in the first column are strictly increasing from top to bottom.*
3. *Consider positive integers $i < j$ such that $(j, k+1) \in \beta /\!\!/ \alpha$. If $(i, k) \in \alpha$ then either $(i, k+1) \in \alpha$ or both $(i, k+1) \in \beta /\!\!/ \alpha$ and $\tau(i, k+1) > \tau(j, k+1)$. If $(i, k) \in \beta /\!\!/ \alpha$ and $\tau(i, k) \geq \tau(j, k+1)$ then either $(i, k+1) \notin \beta /\!\!/ \alpha$ or $\tau(i, k+1) > \tau(j, k+1)$.*

A *standard reverse composition tableau* (SRCT) τ of shape $\alpha \vDash n$ is an SSRCT that contains every positive integer in $[n]$ exactly once.

Throughout this article, when considering an SSRCT, the boxes of the outer shape will be highlighted. Suppose now that we fill the boxes belonging to the inner shape with bullets and assume further that the ordered pairs of positive integers (i, j) such that $(i, j) \notin \alpha$

and $(i,j) \notin \beta$ are filled with 0s. This given, the third condition in the definition above is equivalent to the nonexistence of the following configurations in the filling τ.



$z \geq y > 0$ $\qquad$ $z > 0$ $\qquad$ $x \geq z \geq y > 0$ $\qquad$ $x \geq z > 0$

The existence of a configuration of the above types in a filling will be termed a *triple rule violation*. From this point on, we will refer to the entry in the box in position (i,j) of an SSRCT τ by $\tau_{(i,j)}$.

Example 3.4. An SSRCT of skew reverse composition shape $(2,5,4,2)/\!\!/(1,2,2)$ (left) and an SRCT of reverse composition shape $(3,4,2,3)$ (right).

4	3			
	7	7	5	1
		6	2	

5	4	2	
9	7	6	3
10	1		
12	11	8	

One more notion that we will use is that of the *descent set* of an SRCT. Given an SRCT τ of shape $\alpha \vDash n$, its descent set, denoted by $\mathrm{Des}(\tau)$, is defined to be the set of all integers i such that $i+1$ lies weakly to the right of i in τ. Note that $\mathrm{Des}(\tau)$ is clearly a subset of $[n-1]$. The *descent composition* of τ, denoted by $\mathrm{comp}(\tau)$, is the composition of n associated with $\mathrm{Des}(\tau)$. As an example, the descent set of the SRCT in Example 3.4 is $\{1,2,5,7,9,10\}$. Hence the associated descent composition is $(1,1,3,2,2,1,2)$.

3.3. **Noncommutative symmetric functions.** An algebra closely related to Λ is the algebra of *noncommutative symmetric functions* **NSym**, introduced in [4]. It is the free associative algebra $\mathbb{Q}\langle \mathbf{h}_1, \mathbf{h}_2, \ldots \rangle$ generated by a countably infinite number of indeterminates $\mathbf{h}_k$ for $k \geq 1$. Assigning degree k to $\mathbf{h}_k$, and extending this multiplicatively allows us to endow **NSym** with a structure of a graded algebra. A natural basis for the degree n graded component of **NSym**, denoted by $\mathbf{NSym}^n$, is given by the *noncommutative complete homogeneous symmetric functions*, $\{\mathbf{h}_\alpha = \mathbf{h}_{\alpha_1} \cdots \mathbf{h}_{\alpha_k} \mid \alpha = (\alpha_1, \ldots, \alpha_k) \vDash n\}$. The link between Λ and **NSym** is made manifest through the *forgetful* map, $\chi : \mathbf{NSym} \to \Lambda$, defined by mapping $\mathbf{h}_i$ to h_i and extending multiplicatively. Thus, the images of elements of **NSym** under χ are elements of Λ, imparting credibility to the term noncommutative symmetric function.

NSym has another important basis called the noncommutative ribbon Schur basis. We will denote the *noncommutative ribbon Schur function* indexed by a composition β by $\mathbf{r}_\beta$. The following [4, Proposition 4.13] can be taken as the definition of noncommutative ribbon Schur functions.

$$\mathbf{r}_\beta \quad = \quad \sum_{\alpha \succcurlyeq \beta} (-1)^{l(\beta)-l(\alpha)} \mathbf{h}_\alpha$$

A multiplication rule for noncommutative ribbon Schur functions was proved in [4], and we will be needing it later.

Theorem 3.5. [4, Proposition 3.13] *Let* $\alpha = (\alpha_1, \dots, \alpha_{k_1})$ *and* $\beta = (\beta_1, \dots, \beta_{k_2})$ *be compositions. Define two new compositions* γ *and* μ *as follows*

$$\gamma = (\alpha_1, \dots, \alpha_{k_1}, \beta_1, \dots, \beta_{k_2}),\ \ \mu = (\alpha_1, \dots, \alpha_{k_1} + \beta_1, \beta_2, \dots, \beta_{k_2}).$$

Then

$$\mathbf{r}_\alpha \cdot \mathbf{r}_\beta = \mathbf{r}_\gamma + \mathbf{r}_\mu.$$

We will also need a noncommutative analogue of power sum symmetric functions. In [4], they have actually defined two such analogues. Our interest is in the Ψ basis. For a positive integer n, define

$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k)}. \tag{3}$$

This given, for a composition $\alpha = (\alpha_1, \dots, \alpha_k)$, we define Ψ_α multiplicatively as $\Psi_{\alpha_1} \cdots \Psi_{\alpha_k}$. It can be shown that $\chi(\Psi_n) = p_n$.

While the transition matrix between the Ψ basis and the $\mathbf{r}$ basis is discussed in [4, Section 4.8], no explicit rule for $\Psi_n \cdot \mathbf{r}_\alpha$ is given. Since $\mathbf{r}_\alpha$ has a representation theoretic meaning, this product may also be considered a type of Murnaghan-Nakayama rule and hence, we will give a rule to compute $\Psi_n \cdot \mathbf{r}_\alpha$ where n is a positive integer and α is a composition.

Lemma 3.6. *Let* $\alpha = (\alpha_1, \dots, \alpha_m)$ *be a composition and let* n *be a positive integer. Then*

$$\Psi_n \cdot \mathbf{r}_\alpha = \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k, \alpha_1, \dots, \alpha_m)} + \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k+\alpha_1, \alpha_2, \dots, \alpha_m)}.$$

Furthermore, the summands on the right are all distinct, that is, there is no cancellation in the sum on the right.

Proof. From (3) and the multiplication rule in Theorem 3.5, it follows that

$$\Psi_n \cdot \mathbf{r}_\alpha = \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k, \alpha_1, \dots, \alpha_m)} + \sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k+\alpha_1, \alpha_2, \dots, \alpha_m)}.$$

Hence we only need to establish that the summands above are all distinct. Consider the following multisets of compositions.

$$\begin{aligned} X_1 =& \{(1^k, n-k, \alpha_1, \dots, \alpha_m) \mid 0 \le k \le n-1\} \\ X_2 =& \{(1^k, n-k+\alpha_1, \alpha_2, \dots, \alpha_m) \mid 0 \le k \le n-1\} \end{aligned}$$

By considering the lengths of the compositions therein, it follows that the elements in X_1 (and X_2) are all distinct. Hence, to establish that there is no cancellation, we only need to show that $X_1 \cap X_2$ is empty. Assume otherwise. Then there exist positive integers $0 \le a, b \le n-1$ such that

$$(1^a, n-a, \alpha_1, \dots, \alpha_m) = (1^b, n-b+\alpha_1, \alpha_2, \dots, \alpha_m).$$

That the lengths of the two compositions above have to be equal implies that $a+1=b$. But then the equality of compositions above implies that $n-a=1$, which in turn yields that $b=n$. But we know that $0 \leq b \leq n-1$. Hence we have a contradiction, and the claim follows. ☐

Remark 3.7. As an application we can deduce [4, Proposition 4.23], which gives an expansion of the Ψ basis in terms of the $\mathbf{r}$ basis. While the reader is referred to [4] for the precise details, we briefly describe their result next. Given compositions α and β of the same size, the authors of [4] associate a statistic $psr(\alpha, \beta)$ that is either equal to ± 1 or 0. More precisely, this statistic is ± 1 only when the *ribbon decomposition* of α relative to β involves only hooks, and is 0 otherwise. This given, the following equality is the content of [4, Proposition 4.23].

$$\Psi_\beta = \sum_{\alpha \vDash |\beta|} psr(\alpha, \beta)\mathbf{r}_\alpha$$

Now that we have defined a noncommutative analogue of power sum symmetric functions, to state a noncommutative Murnaghan-Nakayama rule, we need to define our analogue of Schur functions in **NSym**.

3.4. **Noncommutative Schur functions.** We will now describe a distinguished basis for **NSym**, introduced in [2], called the basis of *noncommutative Schur functions*. They are naturally indexed by compositions, and the noncommutative Schur function indexed by a composition α will be denoted by $\mathbf{s}_\alpha$. They are defined implicitly using the relation

$$\mathbf{r}_\beta = \sum_{\alpha \vDash |\beta|} d_{\alpha,\beta}\mathbf{s}_\alpha \tag{4}$$

where $d_{\alpha,\beta}$ is the number of SRCTs of shape α and descent composition β.

Remark 3.8. The original definition of noncommutative Schur functions in [2, Section 2.4] utilized the duality between the Hopf algebras of noncommutative symmetric functions and quasisymmetric functions. The basis of noncommutative Schur functions is defined to be the dual basis corresponding to the basis of quasisymmetric Schur functions. The relation (4) is obtained by considering the dual version of the expansion of a quasisymmetric Schur function in terms of the fundamental quasisymmetric functions [6, Theorem 6.2].

The noncommutative Schur function $\mathbf{s}_\alpha$ satisfies the important property [2, Equation 2.12] that

$$\chi(\mathbf{s}_\alpha) = s_{\widetilde{\alpha}}$$

where $\widetilde{\alpha}$ is the partition obtained by rearranging the parts of α in weakly decreasing order. Thus, noncommutative Schur functions are lifts of Schur functions and, in fact, share many properties with them. The interested reader should refer to [2, 9] for an in-depth study of these functions. For our purposes, we will require the noncommutative Pieri rules for noncommutative Schur functions proved in [2], which we state below.

Theorem 3.9. [2, Corollary 3.8] *Given a composition β, we have*

$$\mathbf{s}_{(n)} \cdot \mathbf{s}_\alpha = \sum_\gamma \mathbf{s}_\gamma$$

where the sum on the right runs over all compositions $\gamma >_c \alpha$ such that $|\gamma /\!\!/ \alpha| = n$ and $\gamma /\!\!/ \alpha$ is a horizontal strip. Similarly,

$$\mathbf{s}_{(1^n)} \cdot \mathbf{s}_\alpha = \sum_\gamma \mathbf{s}_\gamma$$

where the sum on the right runs over all compositions $\gamma >_c \alpha$ such that $|\gamma /\!\!/ \alpha| = n$ and $\gamma /\!\!/ \alpha$ is a vertical strip.

Next, we will state an easy equality that will prove useful later, whilst omitting the proof.

Lemma 3.10. *Let $n \geq 0$. Then we have that $\mathbf{s}_{(n)} = \mathbf{r}_{(n)}$ and $\mathbf{s}_{(1^n)} = \mathbf{r}_{(1^n)}$.*

3.5. **Box-adding operators $\mathfrak{t}_i$.** In this subsection, we will define box-adding operators that act on compositions and add a box to a composition in accordance with the Pieri rules stated in Theorem 3.9.

Definition 3.11. *For every positive integer i, given a composition α, $\mathfrak{t}_i(\alpha)$ is defined in the following manner.*

$$\mathfrak{t}_i(\alpha) = \begin{cases} \beta & \text{if } \alpha \lessdot_c \beta \text{ and } \beta /\!\!/ \alpha \text{ is a box in the } i\text{-th column} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from the above definition that $\mathfrak{t}_i(\alpha)$ is nonzero if and only if α has a part equal to $i-1$.

Example 3.12. Let $\alpha = (2,1,3,1)$. Then $\mathfrak{t}_1(\alpha) = (1,2,1,3,1)$, $\mathfrak{t}_2(\alpha) = (2,2,3,1)$ and $\mathfrak{t}_5(\alpha) = 0$.

It can be easily seen that the box-adding operators satisfy the following useful relation.

Lemma 3.13. *Given positive integers i, j such that $|i-j| \geq 2$, we have the following equality as operators on compositions.*

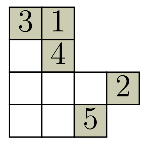
$$\mathfrak{t}_i \mathfrak{t}_j \quad = \quad \mathfrak{t}_j \mathfrak{t}_i$$

3.6. **SRCTs associated with sequence of box-adding operators.** Let α be a composition and suppose $w = \mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_n}$ is such that $w(\alpha) = \beta$, where β is not 0. Then, by [2, Proposition 2.11], we see that there is a unique SRCT of shape $\beta /\!\!/ \alpha$ associated with the word w, as we can associate the following maximal chain in $\mathcal{L}_c$ with w

$$\alpha \lessdot_c \mathfrak{t}_{i_n}(\alpha) \lessdot_c \cdots \lessdot_c \mathfrak{t}_{i_2} \cdots \mathfrak{t}_{i_n}(\alpha) \lessdot_c \mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_n}(\alpha) = \beta.$$

This maximal chain in turn gives rise to a unique SRCT of size n and shape $\beta /\!\!/ \alpha$, where $n-i+1$ is placed in the i-th box added.

Example 3.14. Let $\alpha = (1,3,2)$ and $\beta = \mathfrak{t}_2\mathfrak{t}_4\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3(\alpha) = (2,2,4,3)$. Then $w = \mathfrak{t}_2\mathfrak{t}_4\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3$ uniquely corresponds to the following SRCT.



3.7. **Pieri rules using box-adding operators.** Let $\mathbb{C}Comp$ denote the vector space consisting of formal sums of compositions. Consider the map $\Phi : \mathbf{NSym} \longrightarrow \mathbb{C}Comp$ defined by sending $\mathbf{s}_\alpha \mapsto \alpha$ and extending linearly. We give $\mathbb{C}Comp$ an algebra structure by defining the product between two composition $\alpha \cdot \beta$ as follows.

$$\alpha \cdot \beta \quad = \quad \Phi(\Phi^{-1}(\alpha) \cdot \Phi^{-1}(\beta))$$

It is easily seen that with the above definition of Φ, we have an $\mathbb{C}$-algebra isomorphism between **NSym** and $\mathbb{C}Comp$. We will freely use the notation $\alpha \cdot \beta$ to denote the product $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$, and for the sake of convenience, we will not distinguish between $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$ and $\Phi(\mathbf{s}_\alpha \cdot \mathbf{s}_\beta)$.

With the above setup, we restate Theorem 3.9 in the language of box-adding operators.

Proposition 3.15. *Let α be a composition and n be a positive integer. Then*

$$(n) \cdot \alpha \quad = \quad \left(\sum_{i_n > \cdots > i_1} \mathfrak{t}_{i_n} \cdots \mathfrak{t}_{i_1} \right)(\alpha),$$

$$(1^n) \cdot \alpha \quad = \quad \left(\sum_{i_n \leq \cdots \leq i_1} \mathfrak{t}_{i_n} \cdots \mathfrak{t}_{i_1} \right)(\alpha).$$

In view of Equation (3), we would like to compute $\mathbf{r}_{(1^k, n-k)} \cdot \mathbf{s}_\alpha$ using box-adding operators to obtain a noncommutative analogue of the Murnaghan-Nakayama rule. To this end, we will need to set some more notation.

3.8. **Reverse hookwords and multiplication by $\mathbf{r}_{(1^k,n-k)}$.** Given positive integers n and k such that $0 \leq k \leq n-1$, define $w = \mathfrak{t}_{i_1} \ldots \mathfrak{t}_{i_n}$ to be a *reverse k-hookword* if $i_1 \leq i_2 \leq \cdots \leq i_{k+1} > i_{k+2} > \cdots > i_n$. If w is a reverse k-hookword for some k, then we will call w a *reverse hookword*. Denote by $\mathrm{supp}(w)$ the set formed by the indices $i_1, \ldots, i_n$ (by discarding duplicates). Call w *connected* if $\mathrm{supp}(w)$ is an interval in $\mathbb{Z}_{\geq 1}$, otherwise call it *disconnected*. Put differently, w is connected if the set of indices, when considered in ascending (or descending) order, form a contiguous block of positive integers. Let

$$\mathrm{RHW}_n = \text{set of reverse hookwords of length } n,$$
$$\mathrm{CRHW}_n = \text{set of connected reverse hookwords of length } n.$$

Let $\mathrm{RHW}_{n,k}$ denote the subset of RHW_n consisting of reverse k-hookwords.

For the same w as before, we define the *content vector* of w to be the finite ordered list of nonnegative integers $(c_1, c_2, \ldots)$ where c_i denotes the number of times the operator $\mathfrak{t}_i$ appears in w. Also associated with w are the notions of $\mathrm{arm}(w)$ and $\mathrm{leg}(w)$ defined as follows.

$$\begin{aligned}\mathrm{arm}(w) &= \{i_j \mid 1 \leq j \leq k+1\}\\ \mathrm{leg}(w) &= \{i_j \mid k+1 \leq j \leq n\}\end{aligned}$$

We note that $\mathrm{arm}(w)$ is a multiset, while $\mathrm{leg}(w)$ is a set. Finally, let

$$\mathrm{asc}(w) = |\mathrm{arm}(w)| - 1 = n - |\mathrm{leg}(w)|. \tag{5}$$

Example 3.16. Let $w = \mathfrak{t}_2\mathfrak{t}_5\mathfrak{t}_6\mathfrak{t}_7\mathfrak{t}_8\mathfrak{t}_8\mathfrak{t}_9\mathfrak{t}_9\mathfrak{t}_7\mathfrak{t}_6\mathfrak{t}_4\mathfrak{t}_3$. Then w is a connected reverse 7-hookword with content vector given by $(0,1,1,1,1,2,2,2,2)$. Furthermore, we also have that

$$\begin{aligned}\mathrm{arm}(w) &= \{9,9,8,8,7,6,5,2\},\\ \mathrm{leg}(w) &= \{3,4,6,7,9\}.\end{aligned}$$

Now, using reverse hookwords, we can express the multiplication of noncommutative Schur functions by noncommutative ribbon Schur functions in terms of box-adding operators.

Lemma 3.17. *Let $0 \leq k \leq n-1$, and let α be a composition. Then*

$$\mathbf{r}_{(1^k,n-k)} \cdot \mathbf{s}_\alpha = \left(\sum_{w \in \mathrm{RHW}_{n,k}} w\right)(\alpha).$$

Proof. Note first that in the case $k=0$, the claim is true as it is equivalent to the first identity of Proposition 3.15. We will establish the claim by induction on k. Assume that the claims holds for all integers $\leq k$ for some $k \geq 1$. We will compute $(\mathbf{r}_{(1^{k+1})} \cdot \mathbf{r}_{(n-k-1)}) \cdot \mathbf{s}_\alpha$ in two ways. First notice that by using Theorem 3.5, we get

$$(\mathbf{r}_{(1^{k+1})} \cdot \mathbf{r}_{(n-k-1)}) \cdot \mathbf{s}_\alpha = \mathbf{r}_{(1^{k+1},n-k-1)} \cdot \mathbf{s}_\alpha + \mathbf{r}_{(1^k,n-k)} \cdot \mathbf{s}_\alpha. \tag{6}$$

Now, using Proposition 3.15, we have that

$$\begin{aligned}(\mathbf{r}_{(1^{k+1})} \cdot \mathbf{r}_{(n-k-1)}) \cdot \mathbf{s}_\alpha &= \left(\sum_{i_1 \leq \cdots \leq i_{k+1}} \mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_{k+1}}\right)\left(\sum_{i_{k+2} > \cdots > i_n} \mathfrak{t}_{i_{k+2}} \cdots \mathfrak{t}_{i_n}\right)(\alpha)\\ &= \left(\sum_{w \in \mathrm{RHW}_{n,k}} w\right)(\alpha) + \left(\sum_{w' \in \mathrm{RHW}_{n,k+1}} w'\right)(\alpha),\end{aligned} \tag{7}$$

where the first and second summands correspond to the cases $i_{k+1} > i_{k+2}$ and $i_{k+1} \leq i_{k+2}$ respectively. By the induction hypothesis we know that

$$\mathbf{r}_{(1^k,n-k)} \cdot \mathbf{s}_\alpha = \left(\sum_{w \in \mathrm{RHW}_{n,k}} w\right)(\alpha). \tag{8}$$

Now, using (6), (7) and (8), the claim follows. ∎

4. Multiplication by noncommutative power sums in terms of box-adding operators

On using the definition (3) of the noncommutative power sum function Ψ_n in terms of noncommutative ribbon Schur functions, and Lemma 3.17 subsequently, we get that

$$\begin{aligned} \Psi_n \cdot \mathbf{s}_\alpha &= \left(\sum_{k=0}^{n-1} (-1)^k \mathbf{r}_{(1^k, n-k)}\right) \cdot \mathbf{s}_\alpha \\ &= \sum_{k=0}^{n-1} (-1)^k \sum_{w \in \mathrm{RHW}_{n,k}} w(\alpha) \\ &= \sum_{w \in \mathrm{RHW}_n} (-1)^{\mathrm{asc}(w)} w(\alpha). \end{aligned} \tag{9}$$

Using the involution described in Section 5 of [3], we can restate (9) so that the sum runs over the set of connected reverse hookwords. For completeness, we give a summary of this involution next. Let w be a disconnected reverse hookword and let its content vector be β. Let j be the smallest integer such that $j+1 \notin \mathrm{supp}(w)$, and j is not the maximum element in $\mathrm{supp}(w)$. Such a j exists since $\mathrm{supp}(w)$ is not an interval. Then, if $j \in \mathrm{leg}(w)$, define $\Theta(w)$ to be the unique reverse hookword with content vector β and $\mathrm{leg}(\Theta(w)) = \mathrm{leg}(w) \setminus \{j\}$. Similarly, if $j \in \mathrm{arm}(w)$ but $j \notin \mathrm{leg}(w)$, then define $\Theta(w)$ to be the unique reverse hookword with content vector β and $\mathrm{leg}(\Theta(w)) = \mathrm{leg}(w) \cup \{j\}$. It is easy to see that Θ is an involution on the set of disconnected reverse hookwords, but more importantly, from Lemma 3.13 it follows that $\Theta(w)(\alpha) = w(\alpha)$. Furthermore, $\mathrm{asc}(w)$ and $\mathrm{asc}(\Theta(w))$ are of opposite parity. The preceding arguments allow us to rewrite (9) as follows.

$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_{w \in \mathrm{CRHW}_n} (-1)^{\mathrm{asc}(w)} w(\alpha) \tag{10}$$

While the equation above is a legitimate way to compute $\Psi_n \cdot \mathbf{s}_\alpha$, it is not cancellation-free. Giving a cancellation-free expansion for $\Psi_n \cdot \mathbf{s}_\alpha$ will be our aim in the next section.

Example 4.1. We will compute $\Psi_2 \cdot \mathbf{s}_\alpha$ where $\alpha = (1,3,2)$. The possible elements of CRHW_2 are $\mathfrak{t}_i\mathfrak{t}_i$, $\mathfrak{t}_i\mathfrak{t}_{i+1}$ and $\mathfrak{t}_{i+1}\mathfrak{t}_i$ for $i \geq 1$. Note also that $\mathfrak{t}_j$ acts on α to give 0 when $j \geq 5$. Thus, we have

$$\begin{aligned} \Psi_2 \cdot \mathbf{s}_{(1,3,2)} &= \sum_{i \geq 1} (\mathfrak{t}_{i+1}\mathfrak{t}_i - \mathfrak{t}_i\mathfrak{t}_{i+1} - \mathfrak{t}_i^2)((1,3,2)) \\ &= (\mathfrak{t}_2\mathfrak{t}_1 + \mathfrak{t}_3\mathfrak{t}_2 + \mathfrak{t}_4\mathfrak{t}_3 + \mathfrak{t}_5\mathfrak{t}_4 - \mathfrak{t}_1\mathfrak{t}_2 - \mathfrak{t}_2\mathfrak{t}_3 - \mathfrak{t}_3\mathfrak{t}_4 - \mathfrak{t}_1\mathfrak{t}_1)((1,3,2)) \\ &= (2,1,3,2) + (3,3,2) + (1,4,3) + (1,5,2) - (1,2,3,2) \\ &\quad - (2,3,3) - (1,4,3) - (1,1,1,3,2) \\ &= (2,1,3,2) + (3,3,2) + (1,5,2) - (1,2,3,2) - (2,3,3) - (1,1,1,3,2). \end{aligned}$$

Note that this example reaffirms our claim that (10) is not cancellation-free. The composition (1,4,3) appeared once with coefficient 1, and once with coefficient -1. Hence it does not appear in the final result.

5. The Murnaghan-Nakayama rule for noncommutative Schur functions

By Equation (10) we know that

$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_\beta k_\beta \mathbf{s}_\beta$$

where

$$k_\beta = \sum_{\substack{w \in \mathrm{CRHW}_n \\ w(\alpha)=\beta}} (-1)^{\mathrm{asc}(w)}.$$

Thus, our aim is to compute k_β given any composition β. The expression above hints that it will be useful to tell, given the reverse composition diagrams of α and β, whether there exists a connected reverse hookword w such that $w(\alpha) = \beta$. To this end, we will introduce the notion of an nc border strip.

5.1. **Nc border strips.** Note that if we are given a β such that there exists a $w \in \mathrm{CRHW}_n$ with $w(\alpha) = \beta$, then $\alpha <_c \beta$ and $|\beta /\!/ \alpha| = n$.

Suppose $\alpha <_c \beta$. Define the *support*, $\mathrm{supp}(\beta /\!/ \alpha)$, as follows.

$$\mathrm{supp}(\beta /\!/ \alpha) = \{j \mid (i,j) \in \beta /\!/ \alpha\}.$$

The skew reverse composition shape $\beta /\!/ \alpha$ whose support is an interval in $\mathbb{Z}_{\geq 1}$ will be called an *interval shape*. Finally, an interval shape $\beta /\!/ \alpha$ will be called an *nc border strip* if it satisfies the following conditions.

(1) If there exist boxes in positions $(i,1)$ and $(i,2)$ in $\beta /\!/ \alpha$, then the box in position $(i,1)$ is the bottommost box in column 1 of $\beta /\!/ \alpha$.
(2) If there exist boxes in positions (i,j) and $(i,j+1)$ in $\beta /\!/ \alpha$ for $j \geq 2$, then the box in position (i,j) is the topmost box in column j of $\beta /\!/ \alpha$.

Given an nc border strip $\beta /\!/ \alpha$, its *height* $\mathrm{ht}(\beta /\!/ \alpha)$ is defined to be one less than the number of rows that contain a nonzero number of boxes.

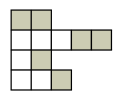
Next, we discuss three notions that have to do with the relative positions of boxes in consecutive columns of an interval shape $\beta /\!/ \alpha$. We say that column $j+1$ is *south-east* of column j in $\beta /\!/ \alpha$ if there is a box in column $j+1$ that is strictly south-east of a box in column j. We say that column $j+1$ is *north-east* of column j, if for any pair of boxes $(i_1, j+1)$ and (i_2, j) in $\beta /\!/ \alpha$, we have $i_1 < i_2$. Here, we assume that there exists at least one box in the $j+1$-th column. Finally, we say that column $j+1$ is *east* of column j if there are boxes in positions (i,j) and $(i,j+1)$ for some i.

This given, we associate three sets with an interval shape $\beta /\!\!/ \alpha$ as follows.

$$\begin{aligned} \mathrm{E}(\beta /\!\!/ \alpha) &= \{j \mid \text{column } j+1 \text{ is east of column } j\} \\ \mathrm{SE}(\beta /\!\!/ \alpha) &= \{j \mid \text{column } j+1 \text{ is south-east of column } j \text{ and } j \notin \mathrm{E}(\beta /\!\!/ \alpha)\ \} \\ \mathrm{NE}(\beta /\!\!/ \alpha) &= \{j \mid \text{column } j+1 \text{ is north-east of column } j\} \end{aligned}$$

We will see later that the statistic $|\mathrm{NE}(\beta /\!\!/ \alpha)|$ plays a crucial role in determining the structure of connected reverse hookwords w satisfying $w(\alpha) = \beta$.

Example 5.1. Consider the following interval shape $\beta /\!\!/ \alpha$, where the unfilled boxes belong to the inner shape and those shaded belong to the outer shape.



In this case we have $\mathrm{E}(\beta /\!\!/ \alpha) = \{1, 4\}, \mathrm{SE}(\beta /\!\!/ \alpha) = \{2\}, \mathrm{NE}(\beta /\!\!/ \alpha) = \{3\}$. Notice that the above interval shape is an nc border strip whose height is 3.

Remark 5.2. Given an interval shape $\beta /\!\!/ \alpha$, note that $\mathrm{E}(\beta /\!\!/ \alpha)$, $\mathrm{SE}(\beta /\!\!/ \alpha)$, $\mathrm{NE}(\beta /\!\!/ \alpha)$ are always pairwise disjoint. Furthermore, if p is the maximum element of $\mathrm{supp}(\beta /\!\!/ \alpha)$, then we have

$$\mathrm{supp}(\beta /\!\!/ \alpha) \quad = \quad \{p\} \uplus \mathrm{E}(\beta /\!\!/ \alpha) \uplus \mathrm{SE}(\beta /\!\!/ \alpha) \uplus \mathrm{NE}(\beta /\!\!/ \alpha).$$

Now, consider the following two sets

$$\begin{aligned} A_{\alpha,n} &= \{\beta \mid \beta = w(\alpha) \text{ for some } w \in \mathrm{CRHW}_n\}, \\ B_{\alpha,n} &= \{\beta \mid \alpha <_c \beta \text{ and } \beta /\!\!/ \alpha \text{ is an nc border strip of size } n\}. \end{aligned}$$

Our aim is to establish that $A_{\alpha,n} = B_{\alpha,n}$. This means that we will be able to replace checking whether there exists $w \in \mathrm{CRHW}_n$ such that $w(\alpha) = \beta$ with checking whether $\beta /\!\!/ \alpha$ is an nc border strip.

5.2. **Equivalence of $A_{\alpha,n}$ and $B_{\alpha,n}$.** We start by establishing the following lemma.

Lemma 5.3. *We have that $B_{\alpha,n} \subseteq A_{\alpha,n}$.*

Proof. Given $\beta \in B_{\alpha,n}$, we will construct an SRCT of shape $\beta /\!\!/ \alpha$ that will be associated (in the sense of Section 3.6) with an element of CRHW_n.

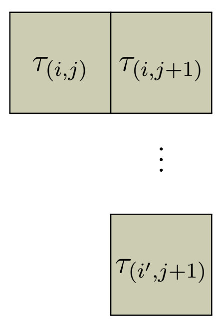
Let $\mathrm{E}(\beta /\!\!/ \alpha) = \{x_1 < \cdots < x_s\}$. For every x_i, place the integer $n - i + 1$ in the topmost box of the x_i-th column of $\beta /\!\!/ \alpha$ if $x_i > 1$ and in the bottommost box if $x_i = 1$. After this step, traverse the remaining $n - s$ unfilled boxes in $\beta /\!\!/ \alpha$ in the manner described below and place the integers $n - s$ down to 1 in that order.

- Start from the rightmost column that contains an empty box.
- Visit every empty box from top to bottom if column under consideration is not the first column, otherwise visit boxes from bottom to top.
- Repeat the above steps for the next rightmost column in $\beta /\!\!/ \alpha$ that contains an empty box.

We claim that the resulting filling, which we call τ, is an SRCT. By construction, if $1 \in \mathrm{supp}(\beta /\!\!/ \alpha)$, then the entries in the boxes in column 1 strictly increase from top to bottom. In all other columns, the entries decrease from top to bottom.

To show that entries decrease from left to right along rows, assume first the existence of boxes in position (i, j) and $(i, j + 1)$ in $\beta /\!\!/ \alpha$. Clearly $j \in \mathrm{E}(\beta /\!\!/ \alpha)$. Since $\beta /\!\!/ \alpha$ is an nc border strip, we know that the box in position (i, j) is the topmost box in the j-th column of $\beta /\!\!/ \alpha$ if $j \geq 2$, and is the bottommost box if $j = 1$. By our construction of the filling, if $j \in \mathrm{E}(\beta /\!\!/ \alpha)$, every box of $\beta /\!\!/ \alpha$ in column $j + 1$ contains a number strictly less than $\tau_{(i,j)}$. Hence in particular, $\tau_{(i,j)} > \tau_{(i,j+1)}$ and thus, the entries decrease from left to right along rows.

Now we have to show that there are no triple rule violations in τ. Since, in every column k where $k \geq 2$, the entries decrease from top to bottom, we are guaranteed there are no triple rule violations of the form



where $j \geq 1$, and $(i', j + 1)$, $(i, j + 1) \in \beta /\!\!/ \alpha$ with $i' > i$.

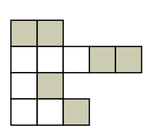
Now assume that $(i, j + 1) \notin \beta /\!\!/ \alpha$. We will show that $\tau_{(i,j)} < \tau_{(i',j+1)}$ for any $i' > i$ where $(i', j + 1) \in \beta /\!\!/ \alpha$. Assume first that $j \in \mathrm{E}(\beta /\!\!/ \alpha)$. If $j = 1$, then we know that (i, j) is not the bottommost box in column j of $\beta /\!\!/ \alpha$, as $(i, j + 1) \notin \beta /\!\!/ \alpha$. Thus, by construction, $\tau_{(i,j)}$ is strictly less than every entry in column 2. In particular, $\tau_{(i',j+1)} > \tau_{(i,j)}$, and we do not have triple rule violations. If $j \geq 2$, then we know that the box in position (i, j) is not the topmost box in column j. Again, we have that $\tau_{(i,j)}$ is strictly less than every entry in column $j + 1$, and as before, we have no triple rule violations.

Now suppose that $(i, j + 1) \notin \beta /\!\!/ \alpha$, and that $j \notin \mathrm{E}(\beta /\!\!/ \alpha)$. Then, the way the filling τ has been constructed implies that the greatest entry in column j is less than the smallest entry in column $j + 1$. This guarantees that $\tau_{(i',j+1)} > \tau_{(i,j)}$, and we have no triple rule violations.

Hence, τ is an SRCT and it is easy to see that the word associated with τ is an element of CRHW_n. Thus, there exists a $w \in \mathrm{CRHW}_n$ such that $w(\alpha) = \beta$, implying $\beta \in A_{\alpha,n}$. ☐

Next, we give an example of the construction presented in the algorithm above.

Example 5.4. Consider the shape $\beta /\!\!/ \alpha$ presented in Example 5.1.



Here $|\beta /\!\!/ \alpha| = 6$. Hence we will be placing the numbers 1 to 6 in the shaded boxes above. Since $\mathrm{E}(\beta /\!\!/ \alpha) = \{1, 4\}$, the bottommost box in column 1 of $\beta /\!\!/ \alpha$ will have 6 placed in it, while the topmost box in column 4 will have 5 placed in it. Now the columns with empty boxes, considered from right to left, are 5, 3 and 2. We fill in these boxes with the numbers $4, 3, 2, 1$ in that order from top to bottom in each column to obtain the following filling.

6	2			
			5	4
	1			
		3		

Lemma 5.5. *Suppose $w \in \mathrm{CRHW}_n$ is such that $w(\alpha) = \beta$. Let τ be the SRCT of shape $\beta /\!\!/ \alpha$ corresponding to w. Then the entries in τ in column j strictly decrease from top to bottom for all $j \geq 2$.*

Proof. Suppose w has c occurrences of $\mathfrak{t}_j$ where $j \geq 2$. We can then factorize w as follows

$$w = w_3 \mathfrak{t}_j^{c-1} w_2 \mathfrak{t}_j w_1,$$

where w_1, w_2 could be empty words. Notice that since w is a reverse hookword, all the operators that constitute w_2 are of the form $\mathfrak{t}_k$ where $k > j$. Hence they only add boxes to parts of length $\geq j$ in the composition $\mathfrak{t}_j w_1(\alpha)$. Now $\mathfrak{t}_j^{c-1}$ adds boxes in the j-th column. Thus, we have added c boxes in the j-th column and the definition of the box-adding operators implies that repeated applications of $\mathfrak{t}_j$ add boxes from top to bottom, when $j \geq 2$. Hence the entries in these c boxes strictly decrease from top to bottom. ☐

Next, we note down a lemma that is an immediate consequence of the bijection mentioned in Subsection 3.6. For Lemma 5.6 and Corollary 5.7, we will work under the assumption that $w \in \mathrm{CRHW}_n$ is such that $w(\alpha) = \beta$, and that τ is the SRCT of shape $\beta /\!\!/ \alpha$ corresponding to w. We will assume further that j belongs to $\mathrm{supp}(\beta /\!\!/ \alpha)$ (or $\mathrm{supp}(w)$), but is not the greatest element therein.

Lemma 5.6.

(1) *If $j \notin \mathrm{leg}(w)$ for some $j \geq 1$, then the greatest entry in column j of τ is strictly less than the smallest entry in column $j + 1$ in τ.*

(2) *If $j \in \mathrm{leg}(w)$ for some $j \geq 1$, then*
- *the greatest entry in column j of τ is strictly greater than than the greatest entry in column $j + 1$ in τ, and*
- *all other entries in column j of τ are strictly smaller than the smallest entry in column $j + 1$ in τ.*

We will note down an important corollary of the lemma above.

Corollary 5.7. *For all $j \geq 1$, every entry in column j except the greatest one, is strictly smaller than the smallest entry in column $j + 1$.*

Now the following lemma, combined with Lemma 5.3, will yield that $A_{\alpha,n} = B_{\alpha,n}$.

Lemma 5.8. *We have that $A_{\alpha,n} \subseteq B_{\alpha,n}$.*

Proof. Assume $\beta \in A_{\alpha,n}$. We have to show that $\beta \in B_{\alpha,n}$ as well. It is clear that if $\beta = w(\alpha)$ for some $w \in \mathrm{CRHW}_n$, then $\alpha <_c \beta$ and $\beta /\!\!/ \alpha$ is an interval shape of size n. Let τ denote the SRCT of shape $\beta /\!\!/ \alpha$ corresponding to w.

Next, suppose there is a configuration of the following type.

$\tau_{(i,j)}$	$\tau_{(i,j+1)}$

If $\tau_{(i,j)}$ is not the greatest entry in column j, then Corollary 5.7 implies that $\tau_{(i,j)} < \tau_{(i,j+1)}$, which contradicts the fact that τ is an SRCT. Hence $\tau_{(i,j)}$ is the greatest entry in column j. Hence the box in position (i, j) is the bottommost box in column j if $j = 1$, or the topmost box in column j otherwise, by invoking Lemma 5.5.

This finishes the proof that $\beta /\!\!/ \alpha$ is an nc border strip if $\beta \in A_{\alpha,n}$. Thus $A_{\alpha,n} \subseteq B_{\alpha,n}$. ☐

We will outline our strategy for the remainder of this section. Now that we have established that $A_{\alpha,n} = B_{\alpha,n}$, our next aim is to enumerate all hookwords w such that $w(\alpha) = \beta$ where $\beta \in B_{\alpha,n}$, that is, $\beta /\!\!/ \alpha$ is an nc border strip of size n. This will be achieved in Lemma 5.12. Once this is done, we will be able to compute the coefficient

$$k_\beta = \sum_{\substack{w \in \mathrm{CRHW}_n \\ w(\alpha)=\beta}} (-1)^{\mathrm{asc}(w)}.$$

Recall that k_β was defined to be the coefficient of $\mathbf{s}_\beta$ in the expansion $\Psi_n \cdot \mathbf{s}_\alpha$. This will give us a noncommutative analogue of the Murnaghan-Nakayama rule that we will state in Theorem 5.16.

5.3. **β-wise enumeration of $w \in \mathrm{CRHW}_n$.** For the rest of this subsection, we will fix an nc border strip $\beta /\!\!/ \alpha$ of size n unless otherwise stated. Also, given this nc border strip $\beta /\!\!/ \alpha$, we define $\mathrm{W}_{\alpha,\beta}$ to be the set of $w \in \mathrm{CRHW}_n$ that satisfy $w(\alpha) = \beta$.

Lemma 5.9. *Let $w \in \mathrm{W}_{\alpha,\beta}$. Then the following hold.*

1. *If column $j \in \mathrm{E}(\beta /\!\!/ \alpha)$, then $j \in \mathrm{leg}(w)$.*
2. *If column $j \in \mathrm{SE}(\beta /\!\!/ \alpha)$, then $j \notin \mathrm{leg}(w)$.*
3. *If $j \in \mathrm{NE}(\beta /\!\!/ \alpha)$, then $j \geq 2$.*

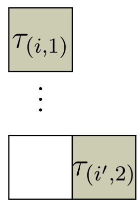
Proof. Let τ be the SRCT of shape $\beta /\!\!/ \alpha$ corresponding to $w = \mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_n}$. We will prove the three parts of the claim above separately.

For the first part, note that if $j \in \mathrm{E}(\beta /\!\!/ \alpha)$ then there exists a configuration in $\beta /\!\!/ \alpha$ of the following form.

$\tau_{(i,j)}$	$\tau_{(i,j+1)}$

We know that $\tau_{(i,j)} > \tau_{(i,j+1)}$, and this implies that there exists $1 \leq p < q \leq n$ such that $\mathfrak{t}_{i_p} = \mathfrak{t}_{j+1}$ and $\mathfrak{t}_{i_q} = \mathfrak{t}_j$. But since w is a reverse hookword, this immediately implies that $j \in \operatorname{leg}(w)$.

Now we proceed to establish the second part. Assume first that $j = 1$. Let the bottommost box in the first column of $\beta /\!\!/ \alpha$ be in position $(i, 1)$. Since $1 \in \mathrm{SE}(\beta /\!\!/ \alpha)$, we know that all boxes in column 2 that belong to $\beta /\!\!/ \alpha$ lie strictly southeast of the box in position $(i, 1)$. Notice that $\tau_{(i,1)}$ is the greatest amongst all the numbers in column 1 of τ. We have the following configuration in τ.



Since τ is an SRCT, we know that $\tau_{(i',2)} > \tau_{(i,1)}$. Thus in particular, the smallest entry in the second column of τ is strictly greater than $\tau_{(i,1)}$. Thus, the first time a box is added in the first column happens after all the boxes that were to be added in the second column by w have been added. Hence $1 \notin \operatorname{leg}(w)$.

Now assume $j \geq 2$. Let $\tau_{(i,j)}$ be the entry in the topmost box of the j-th column of τ. This entry is greater than every other entry in the j-th column of τ by Lemma 5.5. Since $j \in \mathrm{SE}(\beta /\!\!/ \alpha)$, we are guaranteed the existence of a box in position $(i', j+1)$ where $i' > i$. Arguing like before, we must have $\tau_{(i',j+1)} > \tau_{(i,j)}$. Hence there is at least one entry in the $j+1$-th column of τ that is greater than every entry in the j-th column of τ. Given that w is a reverse hookword, this implies that $j \notin \operatorname{leg}(w)$. This concludes the proof of the second part.

Finally, we show that the first column can never belong to $\mathrm{NE}(\beta /\!\!/ \alpha)$. Assume that the claim is not true. Hence $1 \in \mathrm{NE}(\beta /\!\!/ \alpha)$. Thus, we must have that $\{1, 2\} \subseteq \operatorname{supp}(\beta /\!\!/ \alpha)$. If $1 \in \operatorname{leg}(w)$, then the way the box-adding operators act implies that $1 \in \mathrm{E}(\beta /\!\!/ \alpha)$, as $\mathfrak{t}_2$ will add a box adjacent to a newly added box resulting from the operator $\mathfrak{t}_1$ applied before it.

Now, assume $1 \notin \operatorname{leg}(w)$. Then w factors uniquely as $\mathfrak{t}_1^k w_1$, where w_1 has no instances of $\mathfrak{t}_1$. Then we know that in computing $\mathfrak{t}_1^k(w_1(\alpha))$, the boxes added in column 1 will be strictly northwest of the boxes added in column 2 while computing $w_1(\alpha)$. This also follows from how $\mathfrak{t}_1$ acts. But then $1 \in \mathrm{SE}(\beta /\!\!/ \alpha)$. Again, this is a contradiction. Hence we must have that $j \geq 2$. ☐

Lemma 5.10. *Let $j \geq 2$ be a positive integer and μ be a composition. Suppose that μ has parts equaling j and $j-1$ and that the number of parts that equal j and lie to the left of the leftmost instance of a part equaling $j-1$ is m. Then for all $0 \leq k \leq m$, we have*

$$\mathfrak{t}_j \mathfrak{t}_{j+1}^k(\mu) = \mathfrak{t}_{j+1}^k \mathfrak{t}_j(\mu).$$

Proof. The case where $m = 0$ is trivial. Assume that $m \geq 1$. The operator $\mathfrak{t}_{j+1}^k$ adds a box to each of the k leftmost parts in μ that equal j. We know that there are m parts in μ that equal j and lie to the left of the leftmost instance of a part that equals $j-1$. Since $k \le m$, we are guaranteed that $\mathfrak{t}_j \mathfrak{t}_{j+1}^k(\mu) = \mathfrak{t}_{j+1}^k \mathfrak{t}_j(\mu)$. ☐

We will now use the preceding lemma to prove the following result, which is key to obtaining an explicit description for the elements of $\mathrm{W}_{\alpha,\beta}$.

Lemma 5.11. *Let $w \in \mathrm{W}_{\alpha,\beta}$, and let j be such that $j \in \mathrm{NE}(\beta /\!\!/ \alpha)$. Define an element $w' \in \mathrm{CRHW}_n$ as follows.*

- *If $j \in \mathrm{leg}(w)$, then let w' be the unique reverse hookword with the same content as w obtained by setting $\mathrm{leg}(w') = \mathrm{leg}(w) \setminus \{j\}$.*
- *If $j \notin \mathrm{leg}(w)$, then let w' be the unique reverse hookword with the same content as w obtained by setting $\mathrm{leg}(w') = \mathrm{leg}(w) \cup \{j\}$.*

Then $w' \in \mathrm{W}_{\alpha,\beta}$.

Proof. Since $j \in \mathrm{NE}(\beta /\!\!/ \alpha)$, we know that j is not the maximum element of $\mathrm{supp}(\beta /\!\!/ \alpha)$. Thus, the word w' is well-defined. Furthermore, by Lemma 5.9 we know that $j \ge 2$.

Let the number of $\mathfrak{t}_{j+1}$ in w equal c where $c \ge 1$. Let the largest common suffix of w and w' be w_1. All operators $\mathfrak{t}_k$ that appear in w_1 satisfy $k \le j-1$. Let $\mu = w_1(\alpha)$. Since $j \in \mathrm{NE}(\beta /\!\!/ \alpha)$, we know that there are at least c instances of a part of length j to the left of the leftmost part of length $j-1$ in μ. Thus $\mathfrak{t}_j \mathfrak{t}_{j+1}^m(\mu) = \mathfrak{t}_{j+1}^m \mathfrak{t}_j(\mu)$ for $0 \le m \le c$ by Lemma 5.10. This combined with $\mathfrak{t}_j \mathfrak{t}_i = \mathfrak{t}_i \mathfrak{t}_j$ for $|i-j| \ge 2$ implies that $w(\alpha) = w'(\alpha) = \beta$, as required. ☐

Lemma 5.12. *The cardinality of $\mathrm{W}_{\alpha,\beta}$ is $2^{|\mathrm{NE}(\beta /\!\!/ \alpha)|}$, and the elements therein are naturally indexed by subsets of $\mathrm{NE}(\beta /\!\!/ \alpha)$.*

Proof. Let p be the maximum element of $\mathrm{supp}(\beta /\!\!/ \alpha)$. Then, by Remark 5.2, we know that

$$\mathrm{supp}(\beta /\!\!/ \alpha) = \{p\} \uplus \mathrm{E}(\beta /\!\!/ \alpha) \uplus \mathrm{SE}(\beta /\!\!/ \alpha) \uplus \mathrm{NE}(\beta /\!\!/ \alpha). \tag{11}$$

Let $w \in \mathrm{W}_{\alpha,\beta}$. Since the number of instances of $\mathfrak{t}_k$ in w is equal to the number of boxes in column k of $\beta /\!\!/ \alpha$, w is completely determined by $\mathrm{leg}(w)$.

Now by Lemma 5.9, we have that every element of $\mathrm{E}(\beta /\!\!/ \alpha)$ belongs to $\mathrm{leg}(w)$, whereas every element of $\mathrm{SE}(\beta /\!\!/ \alpha)$ does not belong to $\mathrm{leg}(w)$. As far as elements of $\mathrm{NE}(\beta /\!\!/ \alpha)$ are concerned, we know by Lemma 5.11 that it does not matter whether they belong to $\mathrm{leg}(w)$ or not, as the final shape β is going to be the same. Thus, for every subset $X \subseteq \mathrm{NE}(\beta /\!\!/ \alpha)$, the word $w \in \mathrm{CRHW}_n$ satisfying

$$\mathrm{leg}(w) = \mathrm{E}(\beta /\!\!/ \alpha) \uplus X \uplus \{p\} \tag{12}$$

has the property that $w(\alpha) = \beta$. Hence the number of such words is $2^{|\mathrm{NE}(\beta /\!\!/ \alpha)|}$ as claimed. ☐

5.4. **Proof of main theorem - first formulation.** The description for the elements of $\mathrm{W}_{\alpha,\beta}$ obtained earlier implies the following lemma.

Lemma 5.13. *We have the following equality*

$$\sum_{w\in \mathrm{W}_{\alpha,\beta}} (-1)^{\mathrm{asc}(w)} \quad = \quad (-1)^{n-1-|\mathrm{E}(\beta/\!\!/\alpha)|}\delta_{0,|\mathrm{NE}(\beta/\!\!/\alpha)|},$$

where δ denotes the Kronecker delta function.

Proof. Firstly, by Equation (5), we have that $\mathrm{asc}(w) = n - |\mathrm{leg}(w)|$. By Lemma 5.9, we know that $\mathrm{E}(\beta/\!\!/\alpha) \subseteq \mathrm{leg}(w)$ while $\mathrm{SE}(\beta/\!\!/\alpha) \cap \mathrm{leg}(w) = \emptyset$. Furthermore, if p is the maximum element of $\mathrm{supp}(w)$, then $p \in \mathrm{leg}(w)$ as well. Finally, Lemma 5.12 implies that any subset of $\mathrm{NE}(\beta/\!\!/\alpha)$ can belong to $\mathrm{leg}(w)$. Thus, we have the following sequence of equalities.

$$\begin{aligned}
\sum_{w\in \mathrm{W}_{\alpha,\beta}} (-1)^{\mathrm{asc}(w)} &= \sum_{w\in \mathrm{W}_{\alpha,\beta}} (-1)^{n-|\mathrm{leg}(w)|} \\
&= \sum_{X\subseteq \mathrm{NE}(\beta/\!\!/\alpha)} (-1)^{n-1-|\mathrm{E}(\beta/\!\!/\alpha)|-|X|} \\
&= (-1)^{n-1-|\mathrm{E}(\beta/\!\!/\alpha)|} \sum_{X\subseteq \mathrm{NE}(\beta/\!\!/\alpha)} (-1)^{|X|} \\
&= (-1)^{n-1-|\mathrm{E}(\beta/\!\!/\alpha)|}\delta_{0,|\mathrm{NE}(\beta/\!\!/\alpha)|}
\end{aligned}$$

□

What follows is essentially a restatement of the above lemma.

Corollary 5.14. *If $\beta \in B_{\alpha,n}$ then*

$$k_\beta = \begin{cases} 0 & |\mathrm{NE}(\beta/\!\!/\alpha)| \geq 1 \\ (-1)^{n-1-|\mathrm{E}(\beta/\!\!/\alpha)|} & |\mathrm{NE}(\beta/\!\!/\alpha)| = 0. \end{cases}$$

Now, consider the following set of compositions.

$$P_{\alpha,n} = \{\beta \mid \beta \in B_{\alpha,n}, |\mathrm{NE}(\beta/\!\!/\alpha)| = 0\}$$

Before we state a Murnaghan-Nakayama rule in its final definitive form, we will need another short lemma.

Lemma 5.15. *If $\beta/\!\!/\alpha$ is an nc border strip of size n, then $n - 1 - |\mathrm{E}(\beta/\!\!/\alpha)| = \mathrm{ht}(\beta/\!\!/\alpha)$.*

Proof. Suppose there are $k > 0$ boxes in some row of $\beta/\!\!/\alpha$, then this row contributes $k-1$ to $|\mathrm{E}(\beta/\!\!/\alpha)|$. Let the number of boxes in rows that contain at least one box be $k_1, k_2, \ldots, k_r$ for some positive integer r. Notice that $\sum_{j=1}^{r}(k_j - 1)$ equals $n - r$. Thus we have that

$$\begin{aligned}
n - 1 - |\mathrm{E}(\beta/\!\!/\alpha)| &= n - 1 - \sum_{j=1}^{r}(k_j - 1) \\
&= r - 1.
\end{aligned}$$

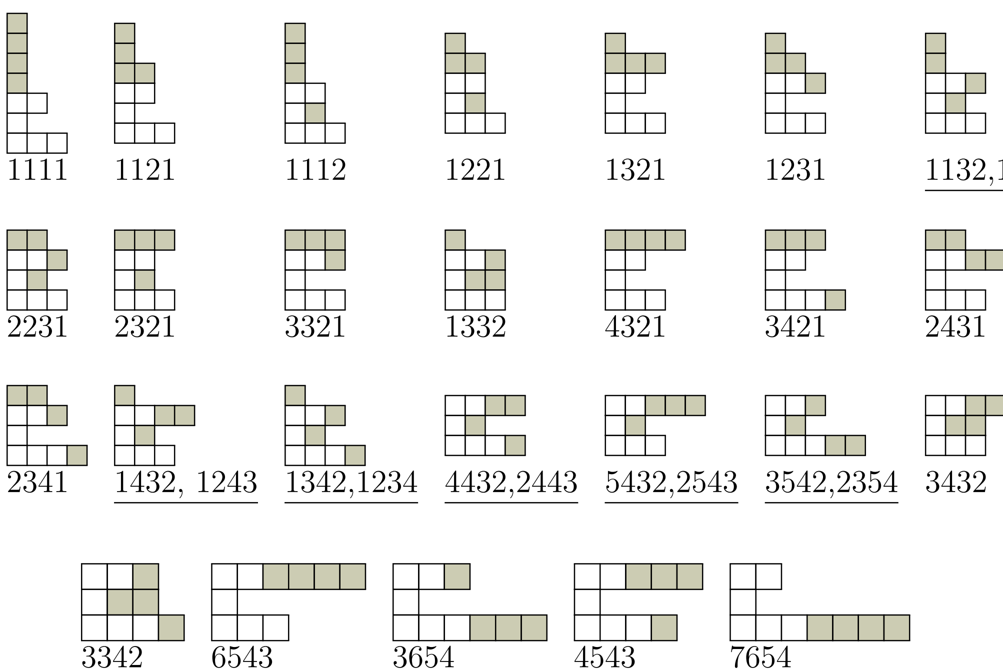
Now notice that $r - 1$ is precisely $\mathrm{ht}(\beta/\!\!/\alpha)$. □

This given, we can write down a Murnaghan-Nakayama rule for noncommutative Schur functions as follows.

Theorem 5.16. *Given a composition α and a positive integer n, we have that*

$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_{\beta \in P_{\alpha,n}} (-1)^{\mathrm{ht}(\beta /\!\!/ \alpha)} \mathbf{s}_\beta.$$

Example 5.17. Consider the computation of $\Psi_4 \cdot \mathbf{s}_{(2,1,3)}$. We need to find all compositions $\beta \in B_{(2,1,3),4}$ satisfying $|\mathrm{NE}(\beta /\!\!/ (2,1,3))| = 0$. We start by listing all the elements of $B_{(2,1,3),4}$ where the boxes of $\beta /\!\!/ \alpha$ will be highlighted. Beneath every composition we have noted those reverse hookwords that act on $(2,1,3)$ to give the composition under consideration. For the sake of clarity, the occurrences of $\mathfrak{t}$ in the reverse hookwords have been suppressed. Thus, for example, 1121 actually denotes $\mathfrak{t}_1\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_1$. Furthermore, if the words below a composition β are underlined, then $|\mathrm{NE}(\beta /\!\!/ (2,1,3)| \geq 1$.



Using Theorem 5.16, we get that the complete expansion is as follows.

$$\begin{aligned}\Psi_4 \cdot \mathbf{s}_{(2,1,3)} &= -\mathbf{s}_{(1,1,1,1,2,1,3)} + \big(\mathbf{s}_{(1,1,2,2,1,3)} - \mathbf{s}_{(1,1,1,2,2,3)}\big) + \mathbf{s}_{(1,2,2,2,3)} + \big(-\mathbf{s}_{(1,3,2,1,3)} + \mathbf{s}_{(1,2,3,1,3)}\big) \\ &\quad + \big(\mathbf{s}_{(2,3,2,3)} - \mathbf{s}_{(3,2,2,3)}\big) + \big(-\mathbf{s}_{(3,3,1,3)} + \mathbf{s}_{(1,3,3,3)}\big) + \big(\mathbf{s}_{(4,2,1,3)} - \mathbf{s}_{(3,2,1,4)} - \mathbf{s}_{(2,4,1,3)} \\ &\quad + \mathbf{s}_{(2,3,1,4)}\big) + \big(-\mathbf{s}_{(4,3,3)} + \mathbf{s}_{(3,3,4)}\big) + \big(\mathbf{s}_{(6,1,3)} - \mathbf{s}_{(3,1,6)}\big) - \mathbf{s}_{(5,1,4)} + \mathbf{s}_{(2,1,7)}\end{aligned}$$

In the example above, we have the written the final expansion in a manner that suggests and motivates the results of the next section. Note first that terms that have been grouped in parentheses are indexed by compositions that have the same underlying partition. As we will see in the next section, the number of terms in every group is always a power of two.

Furthermore, as we shall see in Theorem 6.14 and immediately thereafter, when considering the commutative image of the expansion, the terms within these brackets are precisely the ones that cancel, allowing us to recover the classical expansion.

6. A cancellation-free Murnaghan-Nakayama rule using words

Recall that, by Theorem 5.16, we have the following expansion:

$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_\beta (-1)^{\mathrm{ht}(\beta /\!\!/ \alpha)} \mathbf{s}_\beta,$$

where the sum runs over all compositions β of size $|\alpha| + n$ such that $\beta /\!\!/ \alpha$ is an nc border strip satisfying $|\mathrm{NE}(\beta /\!\!/ \alpha)| = 0$. In the current section we will take an algorithmic approach to compute such compositions β. More specifically, for a fixed partition $\lambda \vdash |\alpha| + n$, we will compute the terms $\mathbf{s}_\beta$ that appear in the expansion above and satisfy $\widetilde{\beta} = \lambda$. But first, we will establish some more notation.

6.1. **Disconnected border strips and valid hookwords.** A skew shape λ/μ is called a *disconnected border strip* if it is a disjoint union of border strips. The number of border strips (or the number of *connected components*) therein will be denoted by $\mathrm{cc}(\lambda/\mu)$. Define its support, $\mathrm{supp}(\lambda/\mu)$, as follows.

$$\mathrm{supp}(\lambda/\mu) = \{j \mid (i,j) \in \lambda/\mu\}$$

Notice that if α is a composition such that $\widetilde{\alpha} = \mu$, and $w \in \mathrm{CRHW}_n$ is such that $\widetilde{w(\alpha)} = \lambda$, then λ/μ is a disconnected border strip of size n whose support is an interval. Equivalently if $\beta /\!\!/ \alpha$ is an nc border strip, then $\widetilde{\beta}/\widetilde{\alpha}$ is a disconnected border strip whose support is an interval. This explains our reasoning for calling $\beta /\!\!/ \alpha$ an nc border strip.

We will now outline our strategy for this section. We will consider a disconnected border strip λ/μ such that $\mathrm{cc}(\lambda/\mu) = k$ and $\mathrm{supp}(\lambda/\mu)$ is an interval. Also let α be a composition such that $\widetilde{\alpha} = \mu$. We will first give a description for the set of all connected reverse hookwords w such that the composition $w(\alpha) = \beta$ satisfies $\widetilde{\beta} = \lambda$. We will denote this set of *valid hookwords* by $\mathrm{VHW}_{\lambda\mu}$. Note that even though the definition of this set depends on α, its elements do not, as will follow from Lemma 6.2. Next, we will filter out the words $w \in \mathrm{VHW}_{\lambda\mu}$ that satisfy $|\mathrm{NE}(w(\alpha) /\!\!/ \alpha)| \geq 1$. The remaining elements of $\mathrm{VHW}_{\lambda\mu}$ will be words w such that $w(\alpha) /\!\!/ \alpha$ is an nc border strip satisfying $|\mathrm{NE}(w(\alpha) /\!\!/ \alpha)| = 0$, and we will then establish that these words contribute distinct compositions β to the expansion in Theorem 5.16.

Our reasons for taking this approach are threefold.

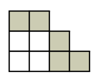
1. Computing disconnected border strips λ/μ is easier than computing nc border strips $\beta /\!\!/ \alpha$.
2. Our approach results in a fairly uniform method to compute coefficients in the product $\Psi_n \cdot \mathbf{s}_\alpha$ for all rearrangements α of μ, as computing $\mathrm{VHW}_{\lambda\mu}$ is the first step for any such computation.

(3) As we will soon establish, the set $\mathrm{VHW}_{\lambda\mu}$ can be endowed with a poset structure that makes it order isomorphic to the Boolean lattice of subsets of a set. The words $w \in \mathrm{VHW}_{\lambda\mu}$ such that $\mathrm{NE}(w(\alpha)/\!\!/\alpha)$ is empty will prove to be a principal order ideal in this poset, and thus we will be able to give an explicit description for such words w.

Finally, we note that using disconnected border strips is natural in this setting as the Murnaghan-Nakayama rules for characters of Iwahori-Hecke algebras [7], and the skew quantum Murnaghan-Nakayama [8] rule do utilize disconnected border strips, even though the classical Murnaghan-Nakayama rule does not.

Let the border strips that constitute λ/μ be labeled $R_1, \cdots, R_k$ from left to right. Furthermore, let l_i (respectively r_i) denote the column number of the left (respectively right) endpoint of the border strip R_i. Also, let $\mathbf{c} = (c_1, \dots, c_{r_k})$, where c_i denotes the number of boxes in column i in λ/μ. Finally, the set of right *endpoints* of the border strips $R_1, \dots, R_{k-1}$ will be denoted by $\mathrm{EP}(\lambda/\mu) = \{r_1, \dots, r_{k-1}\}$. To illustrate the notions introduced in the current subsection, we consider an example next.

Example 6.1. Let $\lambda = (4,3,2)$ and $\mu = (2,2)$. Then λ/μ corresponds to the shaded boxes below.



Clearly λ/μ is a disconnected border strip, and is a disjoint union of two border strips. Hence $\mathrm{cc}(\lambda/\mu) = 2$. The set of left endpoints is $\{1,3\}$ and the set of right endpoints is $\{2,4\}$. Additionally, note that $\mathrm{EP}(\lambda/\mu) = \{2\}$. Finally $\mathbf{c} = (1,1,2,1)$.

The next lemma gives us a description for the words in $\mathrm{VHW}_{\lambda\mu}$. The reader should compare it with Lemma 5.12.

Lemma 6.2. *The distinct elements of* $\mathrm{VHW}_{\lambda\mu}$*, denoted by* w_X*, are indexed by subsets* X *of* $\mathrm{EP}(\lambda/\mu)$*, and* w_X *is defined to be the unique reverse hookword with content vector* $\mathbf{c}$ *and* $\mathrm{leg}(w_X) = (\mathrm{supp}(\lambda/\mu) \setminus \mathrm{EP}(\lambda/\mu)) \uplus X$*. In particular, the cardinality of* $\mathrm{VHW}_{\lambda\mu}$ *is* $2^{\mathrm{cc}(\lambda/\mu)-1}$*.*

Proof. If w is a reverse hookword such that $\widetilde{w(\alpha)} = \lambda$, then by considering how we build $\lambda = \widetilde{w(\alpha)}$ from $\widetilde{\alpha}$ (by going from α to $w(\alpha)$ applying a sequence of box-adding operators, and sorting the composition so as to obtain a partition), we are guaranteed that all elements in $\mathrm{supp}(\lambda/\mu) \setminus \mathrm{EP}(\lambda/\mu)$ belong to $\mathrm{leg}(w)$. Now, for every choice of $X \subseteq \mathrm{EP}(\lambda/\mu)$, we get a unique reverse hookword w with content $\mathbf{c}$ satisfying $\mathrm{leg}(w) = (\mathrm{supp}(\lambda/\mu) \setminus \mathrm{EP}(\lambda/\mu)) \uplus X$. This concludes our description for the elements of $\mathrm{VHW}_{\lambda\mu}$. The claim about the cardinality follows from the fact that $|\mathrm{EP}(\lambda/\mu)| = \mathrm{cc}(\lambda/\mu) - 1$. ☐

Example 6.3. Let $\alpha = (2,2)$ and consider the same skew shape λ/μ as in Example 6.1. Clearly, $\widetilde{\alpha} = \mu$. Note that there are exactly two reverse hookwords that act on α to give a composition β satisfying $\widetilde{\beta} = \lambda$, namely, $\mathfrak{t}_2\mathfrak{t}_3\mathfrak{t}_4\mathfrak{t}_3\mathfrak{t}_1$ and $\mathfrak{t}_3\mathfrak{t}_4\mathfrak{t}_3\mathfrak{t}_2\mathfrak{t}_1$. These two words are all the elements of $\mathrm{VHW}_{\lambda\mu}$, with the former corresponding to $w_\emptyset$ and the latter corresponding to $w_{\{2\}}$.

Remark 6.4. Note that if λ/μ is a border strip, that is, $k = \mathrm{cc}(\lambda/\mu) = 1$, then there is exactly one reverse hookword w such that $\widetilde{w(\alpha)} = \lambda$. Furthermore, it can be seen with a little effort that the connectedness of the border strip implies that $|\mathrm{NE}(w(\alpha)/\!\!/\alpha)| = 0$. Also, the connectedness of individual border strips in a disconnected border strip with $k \geq 2$ components implies that $\mathrm{NE}(w(\alpha)/\!\!/\alpha)$ is always a subset of $\mathrm{EP}(\lambda/\mu)$ for any word $w \in \mathrm{VHW}_{\lambda\mu}$.

In light of the remark above, we assume that $k = \mathrm{cc}(\lambda/\mu) \geq 2$ from now on.

6.2. **A partial order on** $\mathrm{VHW}_{\lambda\mu}$**.** We impose a partial order $\leq_B$ on $\mathrm{VHW}_{\lambda\mu}$ by defining $w_X \leq_B w_Y$ if $Y \subseteq X$. Endowed with this partial order, $\mathrm{VHW}_{\lambda\mu}$ is order isomorphic to the Boolean lattice of subsets of a set of $k-1$ elements (as $\mathrm{EP}(\lambda/\mu)$ has $k-1$ elements), ordered by reverse inclusion.

We now make our strategy for this section even more precise. Let $\mathrm{CVHW}_{\lambda\mu\alpha} \subseteq \mathrm{VHW}_{\lambda\mu}$ comprise of those words w that satisfy $|\mathrm{NE}(w(\alpha)/\!\!/\alpha)| = 0$. We first establish that $\mathrm{CVHW}_{\lambda\mu\alpha}$ is a *principal ideal* in the poset $(\mathrm{VHW}_{\lambda\mu}, \leq_B)$ (provided it is nonempty). More specifically, we establish that there exists a subset $M_{\lambda\mu\alpha}$ of $\mathrm{EP}(\lambda/\mu)$ such that we have the following.

$$w \in \mathrm{CVHW}_{\lambda\mu\alpha} \Longleftrightarrow w \leq_B w_{M_{\lambda\mu\alpha}}$$

Once we prove the above equivalence, we show that for all $w \in \mathrm{CVHW}_{\lambda\mu\alpha}$, the compositions $w(\alpha)$ are all distinct. This allows us to restate Theorem 5.16 in a computationally efficient form.

We begin by defining certain statistics given the composition α, but before that, a cautionary word on our indexing of rows of compositions for this section is in order. If a sequence of box-adding operators appends a new row in front of a composition, then we will refer to this new row as the 0-th row of the resulting composition (instead of calling it the first row). If no new row is appended, then we will not be requiring the notion of the 0-th row.

Now define nonnegative integers γ_{ji} for $1 \leq j \leq i \leq k-1$ as follows.

$$\gamma_{ji} = \begin{cases} \min\{m \mid l_j - 1 \leq \alpha_m \leq r_i - 1\} & l_j > 1 \\ 0 & l_j = 1. \end{cases}$$

Thus, γ_{ji} returns the index of the topmost row of size between $l_j - 1$ and $r_i - 1$ in the composition diagram of α if $l_j > 1$, and returns 0 otherwise. Instead of computing γ_{ji} for all $1 \leq j \leq i$, notice that we only need to compute γ_{ii} for all $1 \leq i \leq k-1$ because of the following fact.

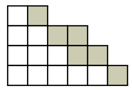
$$\gamma_{ji} = \min\{\gamma_{qq} \mid j \leq q \leq i\}$$

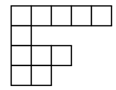
The above immediately implies that

$$\gamma_{1i} \leq \cdots \leq \gamma_{ii}. \tag{13}$$

The relevance of these statistics will become evident soon, but first we give an example.

Example 6.5. Let $\lambda/\mu = (6,5,4,2)/(5,3,2,1)$ (shown highlighted on the left) and $\alpha = (5,1,3,2)$ (shown on the right).





Then we have that $l_1 = 2$, $l_2 = 3$ and $l_3 = 6$ while $r_1 = 2$, $r_2 = 5$ and $r_3 = 6$. This implies that

$$\gamma_{11} = \min\{m \mid 1 \leq \alpha_m \leq 1\} = 2$$
$$\gamma_{12} = \min\{m \mid 1 \leq \alpha_m \leq 4\} = 2$$
$$\gamma_{22} = \min\{m \mid 2 \leq \alpha_m \leq 4\} = 3.$$

Note that $\gamma_{12} = \min\{\gamma_{11}, \gamma_{22}\}$.

Now, given $a \leq b$, we define $\mathfrak{t}_{[a,b]}$ as follows.

$$\mathfrak{t}_{[a,b]} = \mathfrak{t}_b \mathfrak{t}_{b-1} \cdots \mathfrak{t}_{a+1} \mathfrak{t}_a$$

If $a > b$, then $\mathfrak{t}_{[a,b]}$ is understood to be the identity operator on compositions. Armed with this notation, we are ready to state a straightforward lemma.

Lemma 6.6. *Let $w = \mathfrak{t}_{[l_j, r_i-1]}$ where $1 \leq j \leq i \leq k-1$ and let $\beta = w(\alpha)$. Then the γ_{ji}-th row is the topmost row of length equaling $r_i - 1$ in β.*

Proof. This follows from our definition of γ_{ji} and that of the box-adding operators. □

The lemma above has the following crucial implication. Consider r_i for some i satisfying $1 \leq i \leq k-1$, and a word $w \in \mathrm{VHW}_{\lambda\mu}$. Factorize w as

$$w = w_1 w_2$$

where w_2 is the longest suffix of w consisting of all instances of box-adding operators that add boxes in columns strictly left of the r_i-th column. Now consider the composition $w_2(\alpha)$. Then Lemma 6.6 implies that the topmost row of length equaling $r_i - 1$ in $w_2(\alpha)$ is the γ_{ji}-th row for some $1 \leq j \leq i$. In fact, j is the greatest positive integer $\leq i$ such that $r_j \in \mathrm{leg}(w)$ but $r_{j-1} \notin \mathrm{leg}(w)$.

We will be using the argument above repeatedly, without explicit mention.

Lemma 6.7. *Let $X, Y \subseteq \mathrm{EP}(\lambda/\mu)$ be such that $\mathrm{NE}(w_Y(\alpha) /\!\!/ \alpha)$ is empty, and $Y \subseteq X$. Then $\mathrm{NE}(w_X(\alpha) /\!\!/ \alpha)$ is empty as well.*

Proof. We will assume that $X = Y \cup r_i$ for some $1 \leq i \leq k-1$, where $r_i \notin Y$. Once we prove the claim for this choice, the general claim follows by induction.

Suppose, to the contrary, that $\mathrm{NE}(w_X(\alpha) /\!\!/ \alpha)$ is nonempty, and contains r_j for some $1 \leq j \leq k-1$. If $j < i$, then given how w_Y and w_X add boxes and the fact that $X = Y \cup r_i$, it would be the case that $r_j \in \mathrm{NE}(w_Y(\alpha) /\!\!/ \alpha)$. But this would contradict our hypothesis. Hence we must have that $j \geq i$. Let there be b boxes in column $r_j + 1$ in λ/μ.

Consider the following factorization for w_X.

$$w_X = w_1 w_2$$

Here w_2 is the suffix containing all instances of box-adding operators that add boxes in columns strictly left of the r_j-th column, and all b instances of $\mathfrak{t}_{r_j+1}$ belong to w_1. Thus, the topmost row of length $r_j - 1$ in $w_2(\alpha)$ is the γ_{lj}-th row for some $1 \leq l \leq j$. Since $r_j \in \mathrm{NE}(w_X(\alpha) /\!\!/ \alpha)$, we get that there are at least b rows of length r_j above the γ_{lj}-th row in $w_2(\alpha)$.

Now, consider the factorization for w_Y.

$$w_Y = w_3 w_4$$

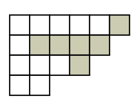
As before, w_4 is the suffix containing all instances of box-adding operators that add boxes in columns strictly left of the r_j-th column, and all b instances of $\mathfrak{t}_{r_j+1}$ belong to w_3. Now, since $r_i \notin Y$ and $i \leq j$, we get that the topmost row of length equaling $r_j - 1$ in $w_4(\alpha)$ is the γ_{mj}-th row where $1 \leq m \leq j$ and $\gamma_{mj} \geq \gamma_{lj}$. Thus, there will be at least b rows of length equaling r_j strictly north of the γ_{mj}-th row in $w_4(\alpha)$. But this implies that r_j will belong to $\mathrm{NE}(w_Y(\alpha) /\!\!/ \alpha)$ as well. This is in contradiction to our hypothesis, and the claim follows. $\square$

Note that the above lemma implies that $\mathrm{CVHW}_{\lambda\mu\alpha}$ is a lower order ideal in $\mathrm{VHW}_{\lambda\mu}$. Furthermore, it also implies the following important equivalence.

$$\mathrm{CVHW}_{\lambda\mu\alpha} \text{ is nonempty} \Longleftrightarrow \mathrm{NE}(w_{\mathrm{EP}(\lambda/\mu)}(\alpha) /\!\!/ \alpha) \text{ is empty.}$$

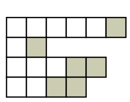
Thus, if $\mathrm{NE}(w_{\mathrm{EP}(\lambda/\mu)}(\alpha) /\!\!/ \alpha)$ is nonempty, then none of the words $w \in \mathrm{VHW}_{\lambda\mu}$ contribute to the expansion in Theorem 5.16. A triple (λ, μ, α) that satisfies $|\mathrm{NE}(w_{\mathrm{EP}(\lambda/\mu)}(\alpha) /\!\!/ \alpha)| \neq 0$ will be called a *bad triple*.

Example 6.8. Consider the skew shape λ/μ and α from Example 6.5. Then $\mathrm{EP}(\lambda/\mu) = \{2, 5\}$, and therefore, $w_{\mathrm{EP}(\lambda/\mu)} = \mathfrak{t}_4\mathfrak{t}_6\mathfrak{t}_5\mathfrak{t}_4\mathfrak{t}_3\mathfrak{t}_2$. Thus $w_{\mathrm{EP}(\lambda/\mu)}(\alpha) /\!\!/ \alpha$ is as shown below.

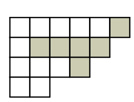


Since $r_2 = 5 \in \mathrm{NE}(w_{\mathrm{EP}(\lambda/\mu)}(\alpha) /\!\!/ \alpha)$ we get that (λ, μ, α) is a bad triple. Below we list the other 3 elements of $\mathrm{VHW}_{\lambda\mu}$ and the resulting compositions when they act on α. The reader can easily verify that $r_2 \in \mathrm{NE}(w(\alpha) /\!\!/ \alpha)$ for all these w.

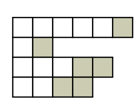
$\mathfrak{t}_2\mathfrak{t}_4\mathfrak{t}_6\mathfrak{t}_5\mathfrak{t}_4\mathfrak{t}_3$



$\mathfrak{t}_4\mathfrak{t}_5\mathfrak{t}_6\mathfrak{t}_4\mathfrak{t}_3\mathfrak{t}_2$



$\mathfrak{t}_2\mathfrak{t}_4\mathfrak{t}_5\mathfrak{t}_6\mathfrak{t}_4\mathfrak{t}_3$



Hence, by our discussion earlier, no term $\mathbf{s}_\delta$ where $\widetilde{\delta} = (6, 5, 4, 2)$ appears in the expansion of $\Psi_6 \cdot \mathbf{s}_{(5,1,3,2)}$.

If $\mathrm{NE}(w_{\mathrm{EP}(\lambda/\mu)}(\alpha)/\!\!/\alpha)$ is empty, then we will call (λ, μ, α) a *good triple*. Since we have already shown that bad triples do not contribute, from now on, we will only consider good triples.

6.3. **Contribution of good triples.** In this subsection we will prove that there is a unique maximal element $w_{M_{\lambda\mu\alpha}}$ such that for all $w \in \mathrm{CVHW}_{\lambda\mu\alpha}$, we have that $w \leq_B w_{M_{\lambda\mu\alpha}}$. This will establish that the lower order ideal $\mathrm{CVHW}_{\lambda\mu\alpha}$ is in fact principal.

Lemma 6.9. *Let $X, Y \subseteq \mathrm{EP}(\lambda/\mu)$ be such that both $\mathrm{NE}(w_X(\alpha)/\!\!/\alpha)$ and $\mathrm{NE}(w_Y(\alpha)/\!\!/\alpha)$ are empty. Then $\mathrm{NE}(w_{Y\cap X}(\alpha)/\!\!/\alpha)$ is empty as well.*

Proof. Let S denote $Y \cap X$. We will proceed by contradiction. Suppose $\mathrm{NE}(w_S(\alpha)/\!\!/\alpha)$ is nonempty and that $r_j \in \mathrm{NE}(w_S(\alpha)/\!\!/\alpha)$ for some $1 \leq j \leq k-1$. Let there be b boxes in column $r_j + 1$ in λ/μ.

Consider the following factorizations for w_X, w_Y and w_S.

$$w_X = w_1 w_2 \qquad\qquad w_Y = w_3 w_4 \qquad\qquad w_S = w_5 w_6$$

Here w_2, w_4 and w_6 consist of all instances of box-adding operators that add boxes in columns strictly less than r_j. Also all b instances of $\mathfrak{t}_{r_j+1}$ in w_X, w_Y and w_S appear in w_1, w_3 and w_5 respectively. Now note that the topmost row of length equaling $r_j - 1$ in $w_2(\alpha)$ is the γ_{lj}-th row for some $1 \leq l \leq j$. Similarly, the topmost row of length $r_j - 1$ in $w_4(\alpha)$ is the γ_{mj}-th row for some $1 \leq m \leq j$. Then, from the fact that $S = Y \cap X$, it follows that the topmost row of length equaling $r_j - 1$ in $w_6(\alpha)$ is the γ_{pj}-th row where $p = \max(l, m)$. Note that in view of (13), this is the same as $\gamma_{pj} = \max(\gamma_{lj}, \gamma_{mj})$. Now, if $r_j \in \mathrm{NE}(w_S(\alpha)/\!\!/\alpha)$, then we know that there are at least b rows of length equaling r_j strictly north of the γ_{pj}-th row. But since $\gamma_{pj} = \max(\gamma_{lj}, \gamma_{mj})$, this would imply that r_j belongs to at least one of $\mathrm{NE}(w_X(\alpha)/\!\!/\alpha)$ and $\mathrm{NE}(w_Y(\alpha)/\!\!/\alpha)$. This is in contradiction to our hypothesis. $\square$

Thus, we have proved in Lemmas 6.7 and 6.9 that $\mathrm{CVHW}_{\lambda\mu\alpha}$ is a lower order ideal in the poset $(\mathrm{VHW}_{\lambda\mu}, \leq_B)$ and it is closed under taking intersections. Hence we can conclude that $\mathrm{CVHW}_{\lambda\mu\alpha}$ is in fact a principal ideal. We note this a corollary next.

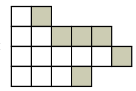
Corollary 6.10. *There is a unique maximal element $w_{M_{\lambda\mu\alpha}}$ in $\mathrm{CVHW}_{\lambda\mu\alpha}$.*

Example 6.11. We will use the skew shape λ/μ from Example 6.5, but this time let $\alpha = (1, 2, 5, 3)$. Note that $\mathrm{EP}(\lambda/\mu) = \{2, 5\}$. Shown below are the compositions resulting by the action of $w_{\{2\}} = \mathfrak{t}_4\mathfrak{t}_5\mathfrak{t}_6\mathfrak{t}_4\mathfrak{t}_3\mathfrak{t}_2$ (left) and $w_{\{5\}} = \mathfrak{t}_2\mathfrak{t}_4\mathfrak{t}_6\mathfrak{t}_5\mathfrak{t}_4\mathfrak{t}_3$ (right) on α .

Since both $\mathrm{NE}(w_{\{2\}}(\alpha)/\!\!/\alpha)$ and $\mathrm{NE}(w_{\{5\}}(\alpha)/\!\!/\alpha)$ are empty, by Lemma 6.9, we have that $\mathrm{NE}(w_{\emptyset}(\alpha)/\!\!/\alpha)$ is empty as well. The computation below (the highlighted boxes showing

$w_\emptyset(\alpha)/\!\!/\alpha)$ reflects this.

$$\mathfrak{t}_2\mathfrak{t}_4\mathfrak{t}_5\mathfrak{t}_6\mathfrak{t}_4\mathfrak{t}_3(\alpha)/\!\!/\alpha =$$



In this case, the maximal element of $\mathrm{CVHW}_{\lambda\mu\alpha}$ is $w_\emptyset$.

Next, we will show that for distinct words $w_1, w_2 \in \mathrm{CVHW}_{\lambda\mu\alpha}$, the compositions $w_1(\alpha)$ and $w_2(\alpha)$ are distinct as well.

Lemma 6.12. *Consider two distinct words $w_1, w_2 \in \mathrm{CVHW}_{\lambda\mu\alpha}$. Then the compositions $w_1(\alpha)$ and $w_2(\alpha)$ are distinct.*

Proof. Consider the smallest integer i such that $r_i \in \mathrm{leg}(w_1)$ but $r_i \notin \mathrm{leg}(w_2)$. We are guaranteed that such an i exists as $w_1 \neq w_2$ by hypothesis. Assume also that the number of boxes in the r_i-th and r_i+1-th column of λ/μ is a and b respectively. Consider the following factorizations of w_1 and w_2:

$$w_1 = w_A \mathfrak{t}_{r_i}^{a-1} w_B \mathfrak{t}_{r_i} w_E \qquad\qquad w_2 = w_C \mathfrak{t}_{r_i}^{a} w_D w_E,$$

where

$$w_E = \mathfrak{t}_{[l_i, r_i-1]}\mathfrak{t}_{[l_{i-1}, r_{i-1}]} \cdots \mathfrak{t}_{[l_1, r_1]}.$$

Note that all b instances of $\mathfrak{t}_{r_i+1}$ in w_1 appear in w_B, while those in w_2 appear in w_D. Also both w_A and w_C add boxes in columns strictly less than r_i.

Suppose first that $r_i = 1$. Then the words w_A and w_C are both empty. We can show easily that the first a parts of $w_2(\alpha)$ have length equaling 1, while in $w_1(\alpha)$ only the first $a-1$ parts have length equaling 1 and the a-th part has length at least 2. Thus, $w_1(\alpha) \neq w_2(\alpha)$ in this situation.

Now assume that $r_i > 1$. Note that the γ_{1i}-th row is the topmost row of size $r_i - 1$ in $w_E(\alpha)$. Since $\mathrm{NE}(w_1(\alpha)/\!\!/\alpha)$ is empty, we infer that in $w_B\mathfrak{t}_{r_i}(w_E(\alpha))$ the γ_{1i}-th row has length at least $r_i + 1$. Similarly, we see that the γ_{1i}-th row in $w_D(w_E(\alpha))$ has length $r_i - 1$. This allows us to conclude that the γ_{1i}-th row in $w_D(w_E(\alpha))$ will grow to be of length r_i in $w_2(\alpha)$. Therefore, we infer that $w_1(\alpha) \neq w_2(\alpha)$. $\square$

We will state another lemma that will prove key when we derive the classical Murnaghan-Nakayama rule from ours.

Lemma 6.13. *If (λ, μ, α) is a good triple and $X = \mathrm{EP}(\lambda/\mu) \setminus \{r_{k-1}\}$, then $\mathrm{NE}(w_X(\alpha)/\!\!/\alpha)$ is empty.*

Proof. Since (λ, μ, α) is a good triple, we know that $\mathrm{NE}(w_{\mathrm{EP}(\lambda/\mu)}(\alpha)/\!\!/\alpha)$ is empty. Let $w_X(\alpha) = \beta$. Assume, contrary to what is to be proved, that $\mathrm{NE}(\beta/\!\!/\alpha)$ is nonempty. But then, comparing how $w_{\mathrm{EP}(\lambda/\mu)}$ and w_X add boxes to α we get that the only element of $\mathrm{NE}(\beta/\!\!/\alpha)$ must be r_{k-1}. But then Lemma 5.11 implies that if w is the word with the same content vector as w_X and such that $\mathrm{leg}(w) = \mathrm{leg}(w_X) \cup \{r_{k-1}\}$, then we have that $w(\alpha) = \beta$. Hence it would follow that $r_{k-1} \in \mathrm{NE}(w(\alpha)/\!\!/\alpha)$. But this word w we constructed here is precisely

$w_{\mathrm{EP}(\lambda/\mu)}$ and the fact that $r_{k-1} \in \mathrm{NE}(w(\alpha)/\!\!/\alpha)$ contradicts the assumption that (λ,μ,α) is a good triple. Hence the claim follows. □

6.4. **Proof of main theorem - second formulation.** Since $(\mathrm{CVHW}_{\lambda\mu\alpha}, \leq_B)$ is a principal ideal in a poset isomorphic to the Boolean lattice, identifying its unique maximal element $w_{M_{\lambda\mu\alpha}}$ is straightforward. We construct $M_{\lambda\mu\alpha}$ as follows: For $1 \leq i \leq k-1$, consider the subset $X_i = \mathrm{EP}(\lambda/\mu) \setminus \{r_i\}$. Then

$$r_i \in M_{\lambda\mu\alpha} \Longleftrightarrow \mathrm{NE}(w_{X_i}(\alpha)/\!\!/\alpha) \text{ is nonempty.}$$

Now we are ready to state the 'algorithmic' version of our noncommutative Murnaghan-Nakayama rule. Unlike Theorem 5.16, this version involves reverse hookwords and should be considered a cancellation-free version of Equation (10). Additionally, this version does not require that nc border strips $\beta/\!\!/\alpha$ satisfying $|\mathrm{NE}(\beta/\!\!/\alpha)| = 0$ be generated by brute force.

Theorem 6.14. *Let n be a positive integer and let α be a composition such that $\widetilde{\alpha} = \mu$. Let BS_1 denote the set of all border strips λ/μ of size n, and let BS_2 be the set of all disconnected border strips λ/μ of size n such that $\mathrm{cc}(\lambda/\mu) \geq 2$ and (λ,μ,α) is a good triple. Then we have the following expansion.*

$$\Psi_n \cdot \mathbf{s}_\alpha = \sum_{\lambda/\mu \in BS_1} \sum_{w \in \mathrm{VHW}_{\lambda\mu}} (-1)^{\mathrm{asc}(w)} \mathbf{s}_{w(\alpha)} + \sum_{\lambda/\mu \in BS_2} \left(\sum_{w \in \mathrm{CVHW}_{\lambda\mu\alpha}} (-1)^{\mathrm{asc}(w)} \mathbf{s}_{w(\alpha)} \right)$$

We will show next how to obtain the classical version (Theorem 2.4) from the expansion above. All we need to establish is that the sum within parentheses is 0. Consider a skew shape $\lambda/\mu \in BS_2$. Then we know that $\mathrm{cc}(\lambda/\mu) \geq 2$ and that (λ,μ,α) is a good triple. Next, consider the following sum.

$$\sum_{w \in \mathrm{CVHW}_{\lambda\mu\alpha}} (-1)^{\mathrm{asc}(w)} \mathbf{s}_{w(\alpha)}$$

For all w over which the sum runs, we know that $\widetilde{w(\alpha)} = \lambda$. Thus the commutative image under χ of the sum above is as follows.

$$\sum_{w \in \mathrm{CVHW}_{\lambda\mu\alpha}} (-1)^{\mathrm{asc}(w)} s_\lambda$$

Therefore to obtain the classical Murnaghan-Nakayama rule from Theorem 6.14, we only need to establish that

$$\sum_{w \in \mathrm{CVHW}_{\lambda\mu\alpha}} (-1)^{\mathrm{asc}(w)} = 0. \tag{14}$$

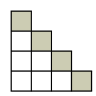
Note that the elements of $\mathrm{CVHW}_{\lambda\mu\alpha}$ are words $w_X \in \mathrm{VHW}_{\lambda\mu}$ where $M_{\lambda\mu\alpha} \subseteq X$. Furthermore, by Lemma 6.13 we know that the set $\mathrm{EP}(\lambda/\mu) \setminus M_{\lambda\mu\alpha}$ is nonempty. Using this fact in an argument very similar to that in Lemma 5.13, the equality in Equation (14) follows. Thus, Theorem 6.14 reduces to Theorem 2.4 when we consider the commutative image.

We also have the following enumerative result (hinted at the end of Section 5) from our description of $\mathrm{CVHW}_{\lambda\mu\alpha}$.

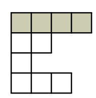
Corollary 6.15. *Let (λ, μ, α) be a good triple, and assume that $|\lambda| - |\mu| = n$. Also, let $X = \mathrm{EP}(\lambda/\mu) \setminus M_{\lambda\mu\alpha}$. Then the number of terms of the form $\pm \mathbf{s}_\beta$ that appear in $\Psi_n \cdot \mathbf{s}_\alpha$ and satisfy $\widetilde{\beta} = \lambda$ is $2^{|X|}$.*

To illustrate the new version of our Murnaghan-Nakayama rule, we will recompute some terms of the product in Example 5.17.

Example 6.16. Consider the expansion of $\Psi_4 \cdot \mathbf{s}_\alpha$ where $\alpha = (2, 1, 3)$. We have that $\mu = \widetilde{\alpha} = (3, 2, 1)$ and let $\lambda = (4, 3, 2, 1)$. We will compute how many terms of the form $\pm \mathbf{s}_\beta$ appear in the expansion where $\widetilde{\beta} = \lambda$. Firstly notice that λ/μ (shown below) is a disconnected border strip with $\mathrm{cc}(\lambda/\mu) = 4$.

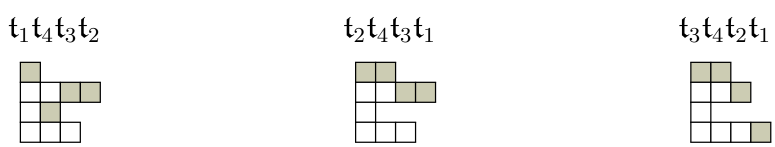


There are 8 elements in $\mathrm{VHW}_{\lambda\mu}$, and $\mathrm{EP}(\lambda/\mu) = \{r_1 = 1, r_2 = 2, r_3 = 3\}$. First we check that (λ, μ, α) is a good triple by computing $w_{\mathrm{EP}(\lambda/\mu)}(\alpha) = \mathfrak{t}_4\mathfrak{t}_3\mathfrak{t}_2\mathfrak{t}_1(\alpha) = (4, 2, 1, 3)$.



Since $\mathrm{NE}(w_{\mathrm{EP}(\lambda/\mu)}(\alpha) /\!\!/ \alpha)$ is empty, we infer that (λ, μ, α) is a good triple.

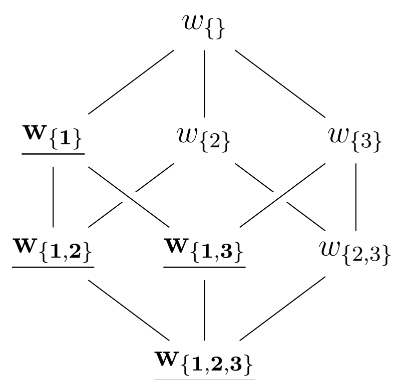
Next we compute $M_{\lambda\mu\alpha}$. Let $X_1 = \{r_2 = 2, r_3 = 3\}$, $X_2 = \{r_1 = 1, r_3 = 3\}$ and $X_3 = \{r_1 = 1, r_2 = 2\}$. We start by computing the action of w_{X_i} on α for $1 \leq i \leq 3$ as shown below.



Since $\mathrm{NE}(w_{X_1}(\alpha) /\!\!/ \alpha)$ is not empty (while $\mathrm{NE}(w_{\mathrm{EP}(\lambda/\mu)}(\alpha) /\!\!/ \alpha)$ is empty), we infer that $r_1 = 1 \in M_{\lambda\mu\alpha}$. Arguing in a similar manner, since both $\mathrm{NE}(w_{X_2}(\alpha) /\!\!/ \alpha)$ and $\mathrm{NE}(w_{X_3}(\alpha) /\!\!/ \alpha)$ are empty, we conclude that $M_{\lambda\mu\alpha} = \{1\}$. Thus, if $w \in \mathrm{VHW}_{\lambda\mu}$ is such that $\mathrm{NE}(w(\alpha) /\!\!/ \alpha)$ is empty, then $1 \in \mathrm{leg}(w)$. There are precisely 4 such words in $\mathrm{VHW}_{\lambda\mu}$. We list them below along with the resulting compositions when they act on α.

Thus, we can conclude that $\Psi_4 \cdot \mathbf{s}_{(2,1,3)} = (\mathbf{s}_{(4,2,1,3)} - \mathbf{s}_{(3,2,1,4)} - \mathbf{s}_{(2,4,1,3)} + \mathbf{s}_{(2,3,1,4)}) + \cdots$.

We display the poset $(\mathrm{VHW}_{\lambda\mu}, \leq_B)$ below, and underline the elements of the order ideal $\mathrm{CVHW}_{\lambda\mu\alpha}$.



Remark 6.17. Given a composition α, let α_1 (repectively α_2) denote the composition obtained by writing the parts in nonincreasing (respectively nondecreasing) order. Then it can be shown that amongst all possible products $\Psi_n \cdot \mathbf{s}_\gamma$ where $\widetilde{\gamma} = \widetilde{\alpha}$, the product $\Psi_n \cdot \mathbf{s}_{\alpha_1}$ has the least number of terms whereas $\Psi_n \cdot \mathbf{s}_{\alpha_2}$ has the most terms.

7. Future work

One possible avenue to consider is to utilize the noncommutativity of **NSym** and find a combinatorial rule to compute $\mathbf{s}_\alpha \cdot \Psi_n$. This will result in a different lift of the classical Murnaghan-Nakayama rule in **NSym**. Data suggest that the coefficients in the expansion in this case are also ± 1. Another avenue is to find a representation-theoretic interpretation for the structure coefficients that occur when we expand a noncommutative power sum symmetric function in terms of noncommutative Schur functions. While we did not address them here, these structure coefficients can be computed by iteratively using our Murnaghan-Nakayama rule. In the classical case, the structure coefficients are character values of the symmetric group, and it would be interesting to know what they are in the setting of **NSym**.

Finally, we must mention the other noncommutative power sum symmetric functions introduced in [4], denoted by Φ_n for n a positive integer. The problem of finding a Murnaghan-Nakayama rule for this basis is still unsolved, and computer experiments suggest that this rule will be substantially more complex than the one considered in this article.

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